

$$g(x) = 3x^2 - x + 5$$

$$\text{Domain} = \mathbb{R} = (-\infty, \infty)$$

$$[3, 5]$$

$$(3, 5] \quad (3, 5]$$

$$\text{---} \circ \text{---} \circ \text{---} \quad \text{---} \circ \text{---} \circ \text{---}$$

$$g(x) = 3x^2 - x + 5$$

$$g(x+h) = 3(x+h)^2 - (x+h) + 5$$

$$(x+h)^2 = x^2 + 2xh + h^2$$

$$= 3(x^2 + 2xh + h^2) - x - h + 5$$

$$= 3x^2 + 6xh + 3h^2 - x - h + 5$$

$$= \dots$$

$$\text{---} \circ \text{---} \circ \text{---} \quad \infty$$

$$\text{Domain: } \mathbb{R} \setminus \{4/3\} \text{ : Set notation}$$

$$\text{'' : } (-\infty, 4/3) \cup (4/3, \infty) \text{ : Interval notation}$$

Let

$$f(x) = \frac{2x}{3x-4}$$

Find the domain of f

Solution: Find the excluded values:

$$3x - 4 = 0$$

$$3x = 4$$

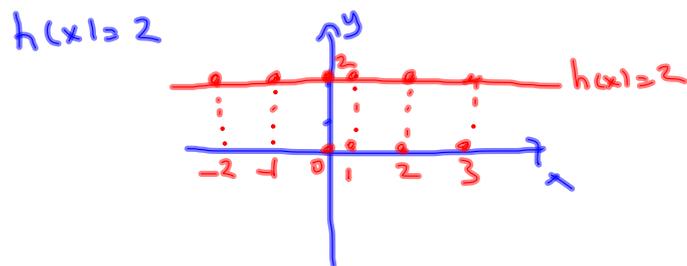
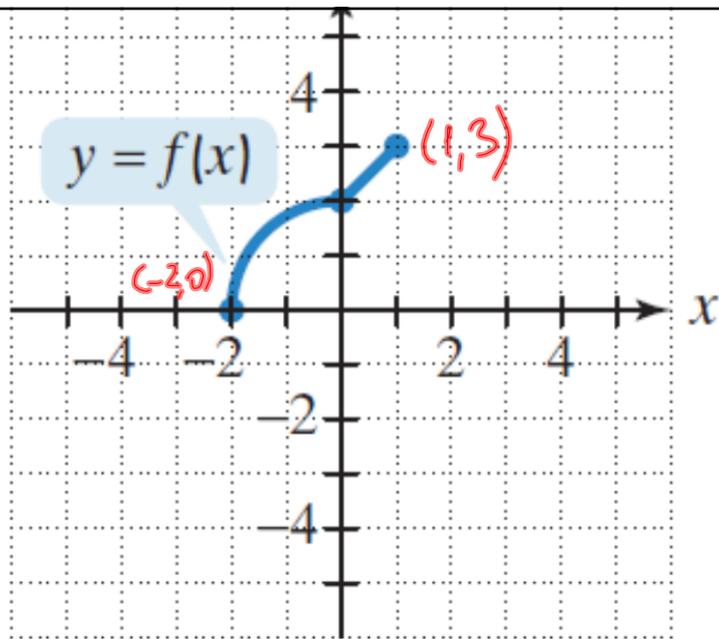
$$x = \frac{4}{3}$$

$$\frac{0}{-4} = 0$$

$$\frac{2 \cdot \frac{4}{3}}{0} = \frac{8/3}{0}$$

Undefined

$$D(f) = \mathbb{R} \setminus \{\frac{4}{3}\} = (-\infty, \frac{4}{3}) \cup (\frac{4}{3}, +\infty)$$



$$|-3| = -1(-3) = 3$$

$$|3| = 3$$

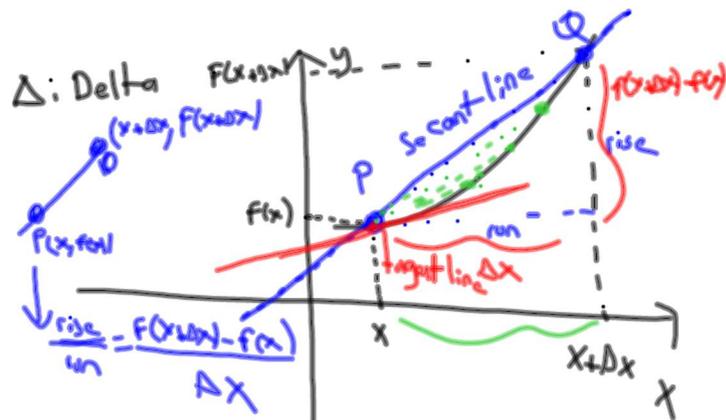
$$|-\pi| = \pi$$

$$|x-1| \text{ if } x < 1$$

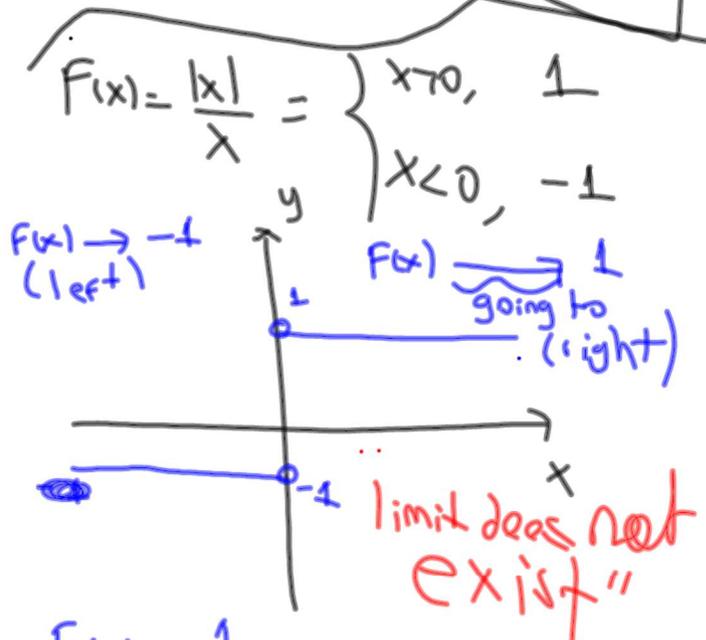
$$|x-1| = -(x-1) = -x+1, \text{ if } x < 1$$

$$|x-1| \text{ if } x > 1$$

$$|x-1| = x-1 \text{ if } x > 1.$$



Slope of tangent line: $\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$



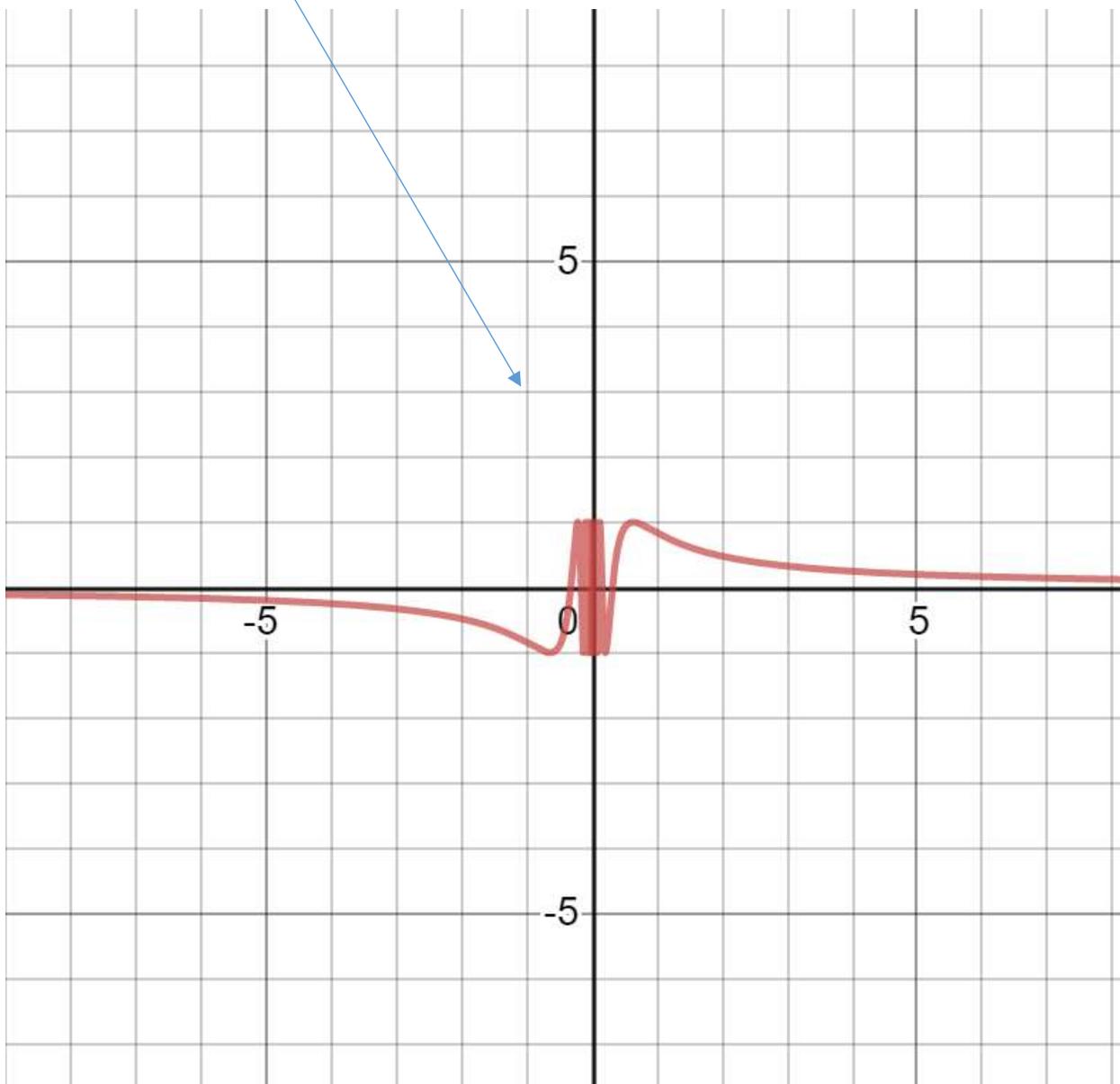
$$f(x) = \frac{1}{x-1}$$

$f(x)$ increases without bound when $x \rightarrow 1$.

$$\lim_{x \rightarrow 1} \frac{x^3 - 1^3}{x - 1} = \frac{(x-1)(x^2 + x + 1)}{x-1} = 3$$

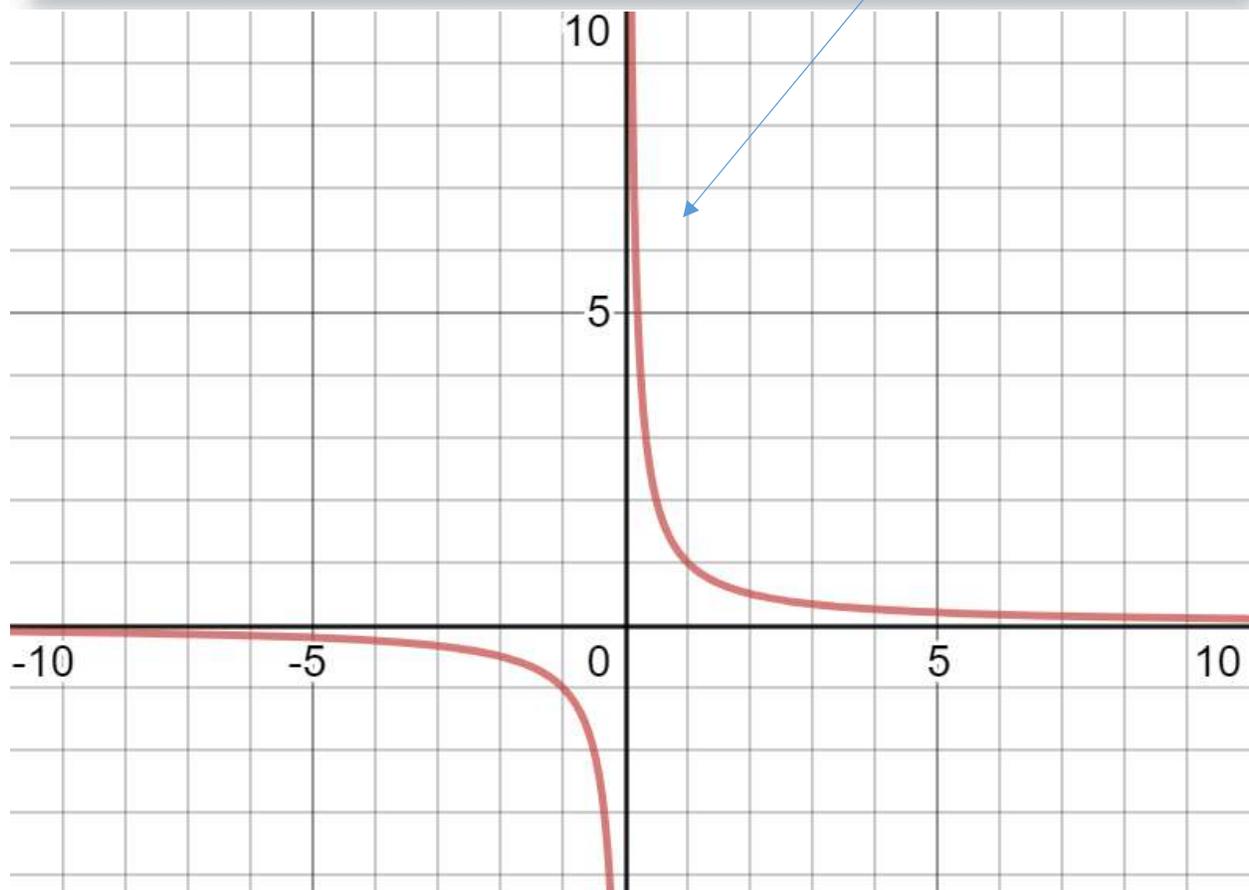
Common Types of Behavior Associated with Nonexistence of a Limit

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as x approaches c .
3. $f(x)$ oscillates between two fixed values as x approaches c .



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Another way to find a limit analytically is the **rationalizing technique**, which involves rationalizing the numerator of a fractional expression.

Recall that rationalizing the numerator means multiplying the numerator and denominator by the **conjugate** of the numerator.

For instance, to rationalize the numerator of $\frac{\sqrt{x+4}}{x}$

multiply the numerator and denominator by the conjugate of $\sqrt{x+4}$, which is $\sqrt{x-4}$.

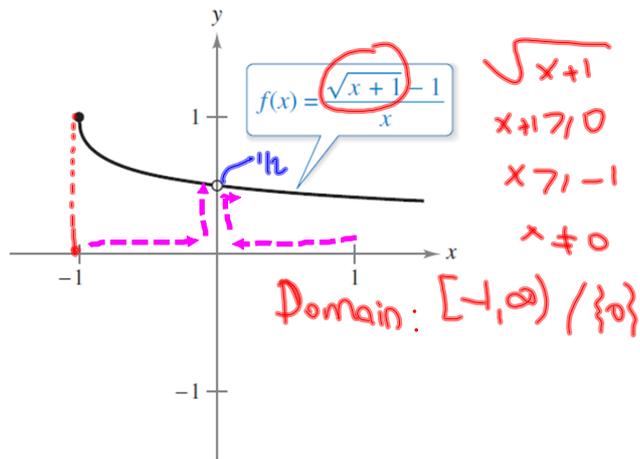
$$\begin{array}{l} a-b \text{ --- } a+b \\ a+b \text{ --- } a-b \\ \sqrt{a+b} \text{ --- } \sqrt{a-b} \\ \sqrt{a-b} \text{ --- } \sqrt{a+b} \end{array} \quad \begin{array}{l} (a+b)(a-b) = a^2 - b^2 \\ (\sqrt{a+b})(\sqrt{a-b}) = a - b^2 \end{array}$$

0/0.

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x}$$

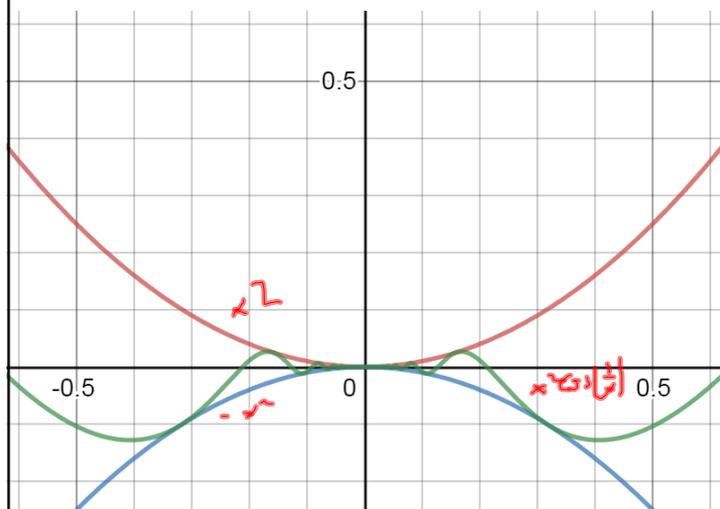
$\lim_{x \rightarrow 0} (\sqrt{x+1} - 1) = 0$
 $\lim_{x \rightarrow 0} x = 0$ (Direct substitution fails.)

$$\frac{\sqrt{x+1} - 1}{x} \cdot \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1} = \frac{x+1-1}{x(\sqrt{x+1}+1)} = \frac{x}{x(\sqrt{x+1}+1)}$$
$$\lim_{x \rightarrow 0} \frac{1}{\sqrt{x+1}+1} = \frac{1}{2} = 0.5$$



The limit of $f(x)$ as x approaches 0 is $\frac{1}{2}$.

Squeeze theorem



Sep 17-09:24

$$\lim_{x \rightarrow 1} \frac{3x - 5}{x^2 + 1} = \frac{-2}{2} = -1 \text{ . Direct Subst.}$$

$$\lim_{x \rightarrow -1} \frac{x^2 + 6x + 5}{x + 1} = \frac{0}{0} \text{ , } \lim_{x \rightarrow -1} \frac{\cancel{(x+1)}(x+5)}{\cancel{x+1}} = 4$$

$$\lim_{x \rightarrow -1} \frac{x^2 + 6x}{x + 1} \text{ ; } \begin{array}{l} x > -1 : -\infty \text{ Right limit} \\ x < -1 : +\infty \text{ Left limit} \end{array}$$



$$\lim_{x \rightarrow 2} \frac{4 - \sqrt{x + 14}}{x - 2}$$

$$\lim_{x \rightarrow 2} f(x), \text{ where } f(x) = \begin{cases} x + 2, & x \leq 2 \\ 6 - x, & x > 2 \end{cases}$$

$f(x) = \frac{x^2-1}{x-1}$ is not continuous at $x=1$.

Because $f(1)$ is undefined

$$\lim_{x \rightarrow 1} \frac{x^2-1}{x-1} = \frac{0}{0} \quad \lim_{x \rightarrow 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = 2$$

$f(1) = 2$. $g(x) = \begin{cases} f(x), & x \neq 1 \\ 2, & x = 1 \end{cases}$

c. $h(x) = \begin{cases} x + 1, & x \leq 0 \\ x^2 + 1, & x > 0 \end{cases}$

$(x \leq 0)$
Left limit at $x=0$: $0 + 1 = 1$.

Right " " " : $0^2 + 1 = 1$.
 $(x > 0)$

Limit exists. $= f(0) = 1$.

f is continuous everywhere (on \mathbb{R})

$$\sqrt{-9} \times \sqrt{-27} = -3$$

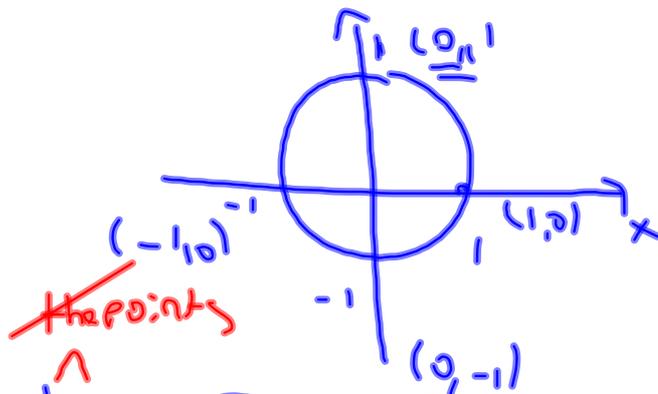
$$f(x) = \sqrt{1-x^2} \quad 1-x^2 \geq 0$$

$$\Rightarrow x^2 \leq 1 \Rightarrow x \in [-1, 1]: \text{Domain}$$

$$\lim_{x \rightarrow 1^-} f(x) = 0 = f(1) \quad \lim_{x \rightarrow -1^+} f(x) = 0 = f(-1)$$

Continuous on $[-1, 1]$

$$f(x) = \tan x = \frac{\sin x}{\cos x}, \quad \cos x \neq 0$$



$$\cos x = 0 \text{ if}$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \dots$$

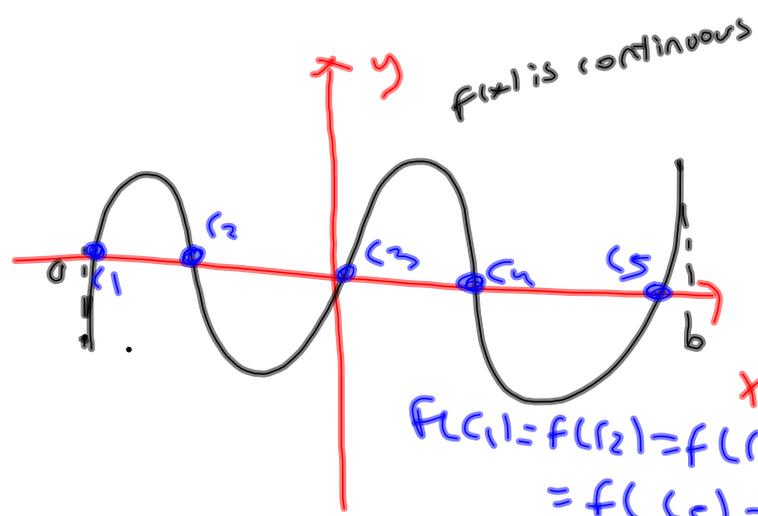
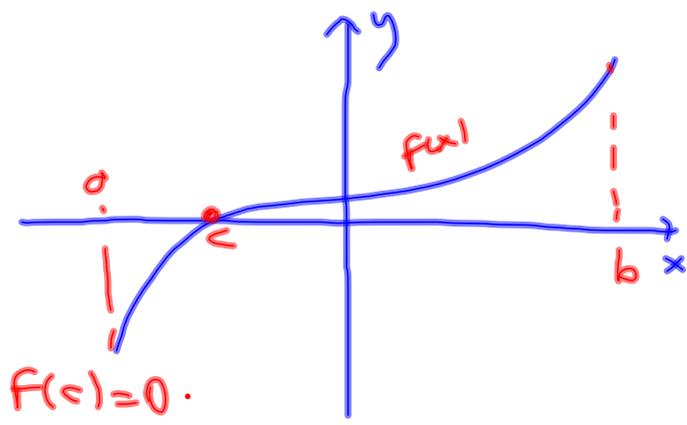
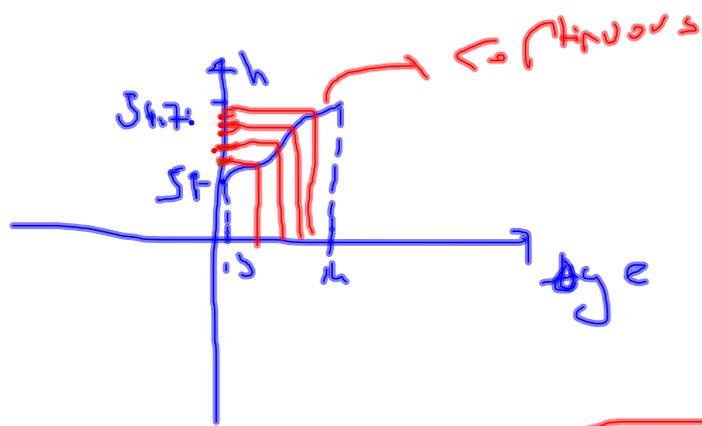
$$x = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}$$

Except $x = \frac{\pi}{2} + n\pi$, $\tan x$ is continuous.

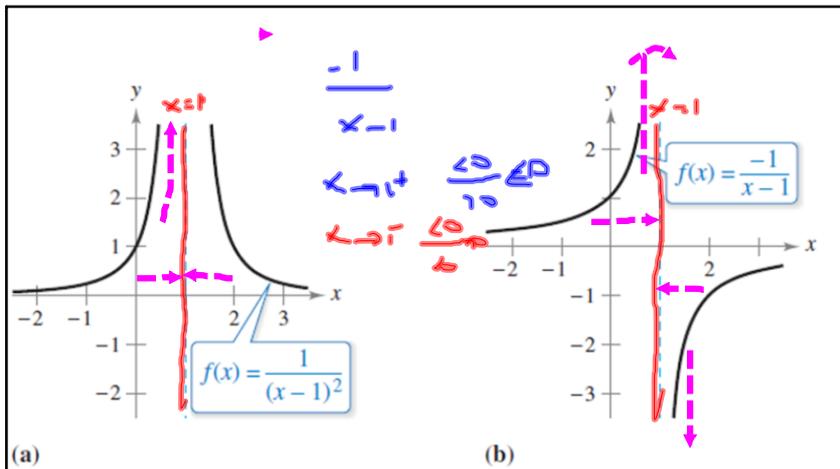
$$\text{b. } g(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$g(x)$ is continuous everywhere except $x = 0$.

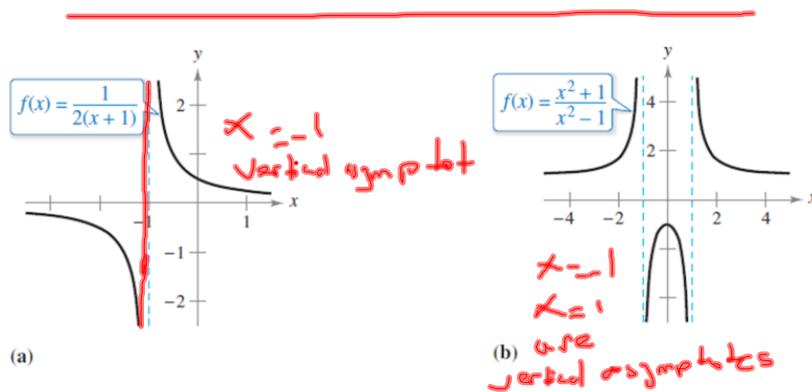
$$\mathbb{R} \setminus \{0\} \\ (-\infty, 0) \cup (0, \infty)$$



$f(c_1) = f(c_2) = f(c_3) = f(c_4) = f(c_5) = 0$
 $f(x) = 0$ has exactly 5 roots.



Each graph has an asymptote at $x = 1$.



Example 2 – Finding Vertical Asymptotes

Determine all vertical asymptotes of the graph of each function.

a. $f(x) = \frac{1}{2(x+1)}$ $\rightarrow x = -1$

b. $f(x) = \frac{x^2+1}{x^2-1}$ $\rightarrow x = -1, x = 1$

c. $f(x) = \cot x$ $\rightarrow \frac{\cos x}{\sin x}$

$x = \dots -4\pi, -3\pi, -2\pi, -\pi, \pi, 2\pi, \dots$

$x = n\pi, n \in \mathbb{Z}$

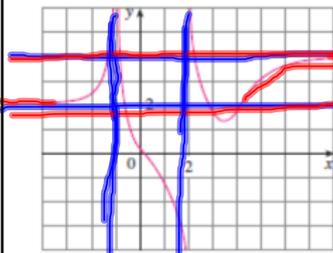


FIGURE 5

$x = -1$ $x = 2$

EXAMPLE 1 Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown in Figure 5.

SOLUTION We see that the values of $f(x)$ become large as $x \rightarrow -1$ from both sides, so

$$\lim_{x \rightarrow -1} f(x) = \infty$$

Notice that $f(x)$ becomes large negative as x approaches 2 from the left, but large positive as x approaches 2 from the right. So

$$\lim_{x \rightarrow 2^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow 2^+} f(x) = \infty$$

Thus both of the lines $x = -1$ and $x = 2$ are vertical asymptotes.

As x becomes large, it appears that $f(x)$ approaches 4. But as x decreases through negative values, $f(x)$ approaches 2. So

$$\lim_{x \rightarrow \infty} f(x) = 4 \quad \text{and} \quad \lim_{x \rightarrow -\infty} f(x) = 2$$

This means that both $y = 4$ and $y = 2$ are horizontal asymptotes. ■

$\lim_{x \rightarrow \infty} f(x) = 4$
 $x \rightarrow -\infty$

EXAMPLE 3 Evaluate

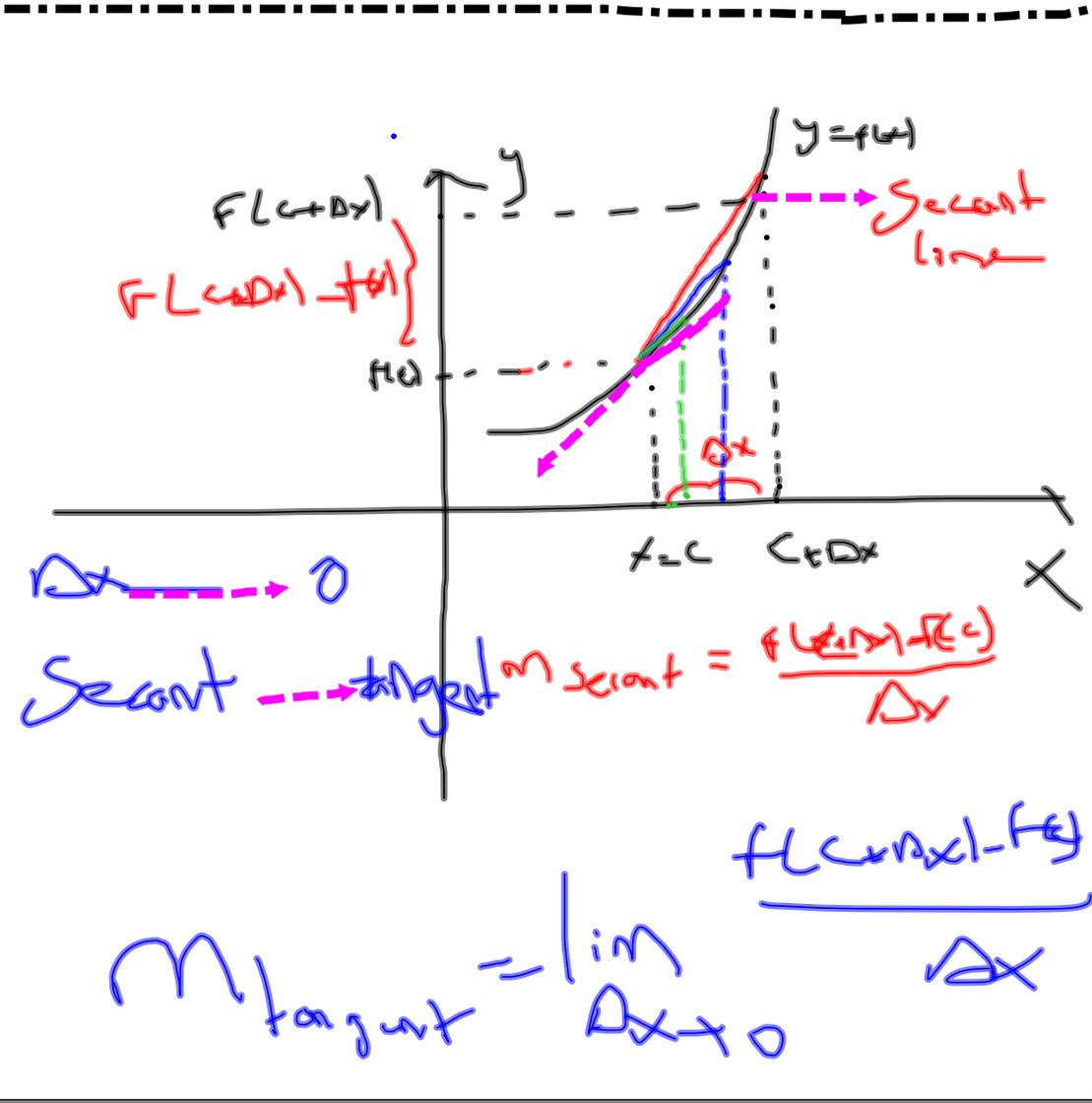
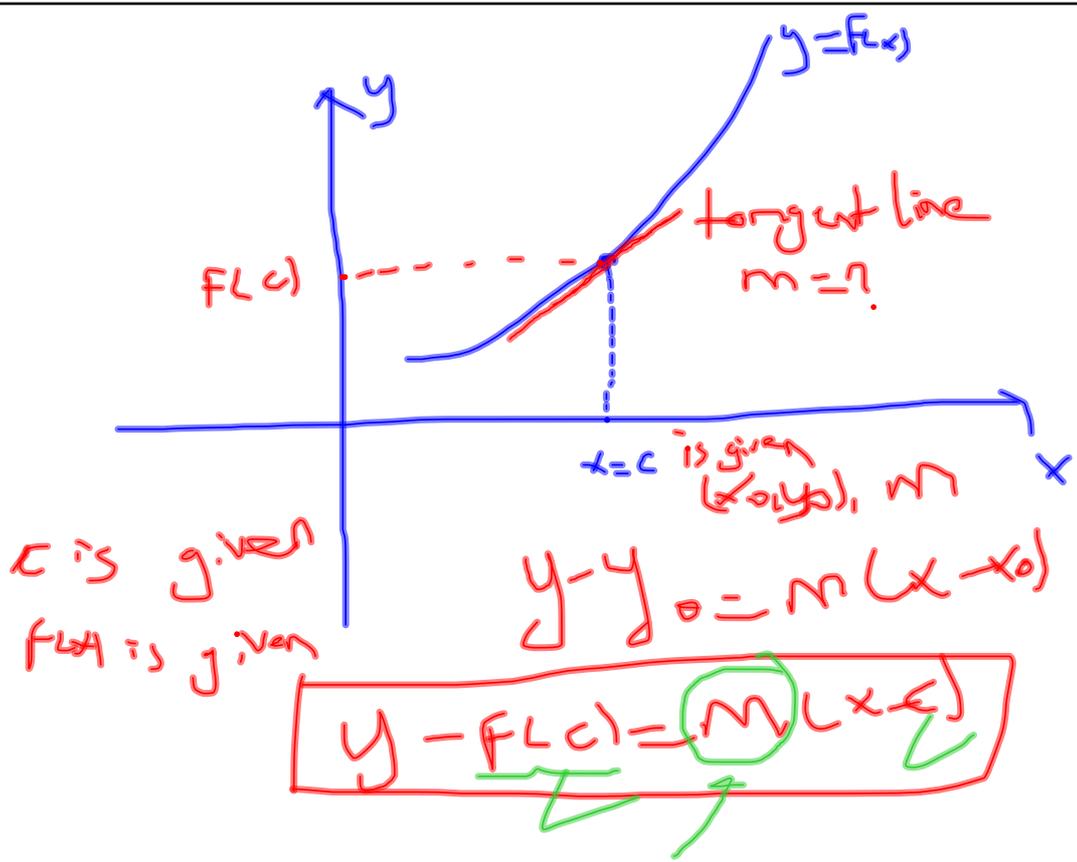
$$\lim_{x \rightarrow \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

and indicate which properties of limits are used at each stage.

SOLUTION As x becomes large, both numerator and denominator become large, so it isn't obvious what happens to their ratio. We need to do some preliminary algebra.

$$\lim_{x \rightarrow \infty} \frac{3x^2 - \frac{x}{x^2} - \frac{2}{x^2}}{5x^2 + \frac{4x}{x^2} + \frac{1}{x^2}} = \frac{3}{5}$$

$$\lim_{x \rightarrow \infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}} = \frac{3}{5}$$



Example 1 – The Slope of the Graph of a Linear Function

$$y = f(x) = 2x - 3$$

To find the slope of the graph of f when $c = 2$, you can apply the definition of the slope of a tangent line, as shown.

$$\begin{aligned}\lim_{\Delta x \rightarrow 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{[2(2 + \Delta x) - 3] - [2(2) - 3]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{4 + 2\Delta x - 3 - 4 + 3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2\Delta x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} 2 \\ &= 2\end{aligned}$$

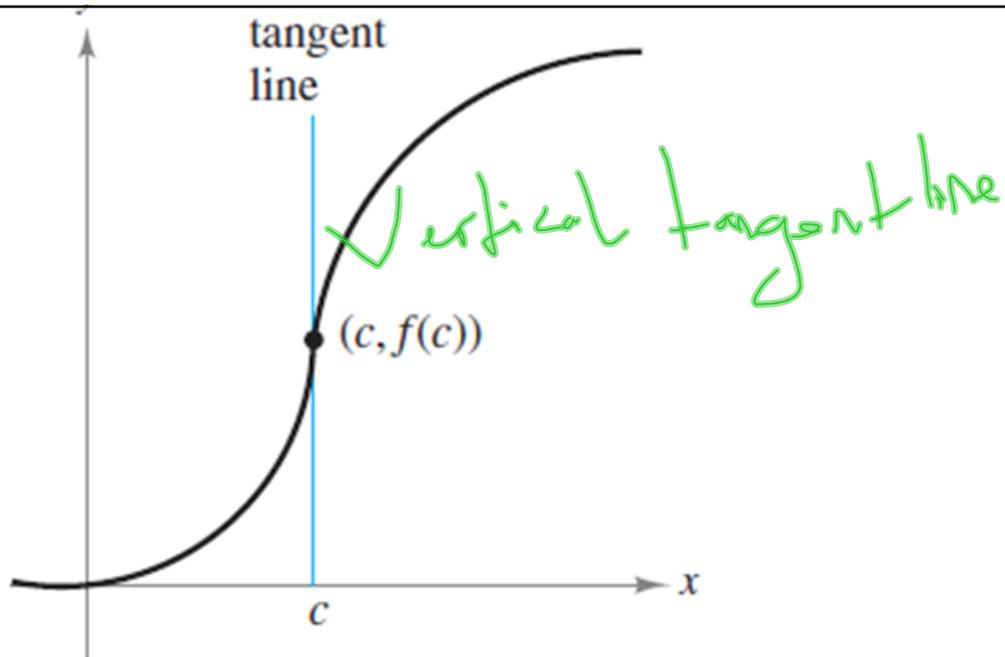
$$y = f(x) = 2x - 3$$

$$\lim_{\Delta x \rightarrow 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{2(2 + \Delta x) - 3 - 1}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{4 + 2\Delta x - 4}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} 2 = 2$$



The graph of f has a vertical tangent line at $(c, f(c))$.



Let $y = f(x)$

$y', f', \frac{dy}{dx}, \frac{df}{dx}$

$f(x) = x^3 + 2x$

$\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$f(x+\Delta x) = (x+\Delta x)^3 + 2(x+\Delta x)$
 $= x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 + 2x + 2\Delta x$

$f(x+\Delta x) - f(x) =$
 ~~$x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 + 2x + 2\Delta x - (x^3 + 2x)$~~

$\lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 + 2\Delta x}{\Delta x}$

$= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x + \Delta x^2 + 2)$

$= 3x^2 + 2 = y' = \frac{dy}{dx} = \frac{df}{dx}$

$y = x^3 + 2x \Rightarrow y' = 3x^2 + 2$

ex. Eq. of the tangent line

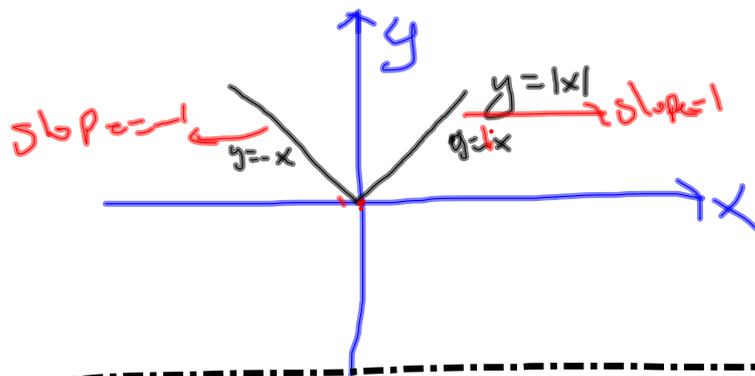
$x_0 = 1$

$y - y_0 = m(x - x_0)$

$y - 3 = 5(x - 1)$

$y = 5x - 5 + 3$

$y = 5x - 2$



$$y = x^3 \Rightarrow y' = 3x^2$$

$$y = x^{-2} \Rightarrow y' = 2 \cdot x^{-3}$$

$$y = \sqrt{x} \Rightarrow y' = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$y = 2x \Rightarrow y' = 2$$

$$y = 3x^{-4} \Rightarrow y' = 3 \cdot \frac{d}{dx} x^{-4} \\ = 3 \cdot -4 \cdot x^{-5} \\ = -12x^{-5}$$

$$y(t) = 3t^{-3} \Rightarrow y' = 3 \cdot -3t^{-4} \\ \frac{dy}{dt} = -9t^{-4}$$

Function

a. $f(x) = x^3 - 4x + 5$

b. $g(x) = -\frac{x^4}{2} + 3x^3 - 2x$

c. $y = \frac{3x^2 - x + 1}{x} = 3x - 1 + \frac{1}{x}$

Derivative

$f'(x) = 3x^2 - 4$

$g'(x) = -2x^3 + 9x^2 - 2$

$y' = 3 - \frac{1}{x^2} = \frac{3x^2 - 1}{x^2}$

a) $F' = 3x^2 - 4 \checkmark$

b) $g = -\frac{1}{2}x^4 + 3x^3 - 2x$
 $g' = -\frac{1}{2} \cdot 4x^3 + 3 \cdot 3x^2 - 2$

$= -2x^3 + 9x^2 - 2$

c) $y = \frac{3x^2 - x + 1}{x}$

$y = \frac{3x^2}{x} - \frac{x}{x} + \frac{1}{x}$

$= 3x - 1 + x^{-1}$

$y' = 3 - 0 + -1x^{-2}$
 $= 3 - x^{-2}$

$= 3 - \frac{1}{x^2}$

$\frac{d}{dx} \left(\frac{a^x}{e} \right) = \ln a$

$\frac{d}{dx} (e^x) = e^x \cdot \frac{\ln e}{1} = e^x$

$\Rightarrow y = e^x \Rightarrow y' = e^x$

$$y = 2^x \Rightarrow y' = 2^x \cdot \ln 2$$

Note: A pink dashed arrow points from the exponent x to the word "Power" written in red above it.

$$y = x^2 \Rightarrow y' = 2x$$

Note: A pink dashed arrow points from the base x to the word "Base" written in red above it.

At time $t = 0$, a diver jumps from a platform diving board that is 32 feet above the water (see Figure 2.21). Because the initial Velocity of the diver is 16 feet per second, the position of the diver is

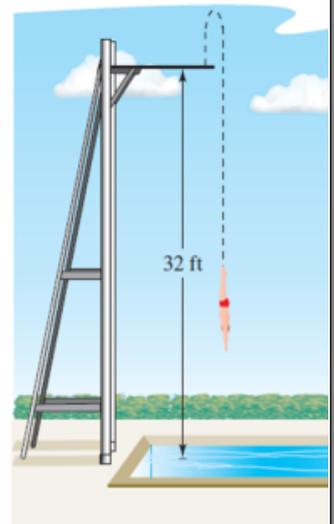
$$s(t) = -16t^2 + 16t + 32 \quad \text{Position function}$$

where s is measured in feet and t is measured in seconds.

- $s(t) = 0 \Rightarrow$ solve for t*
- When does the diver hit the water?
 - What is the diver's velocity at impact?

Find $s'(t_0)$

$t = t_0$



Velocity is positive when an object is rising, and is negative when an object is falling. Notice that the diver moves upward for the first half-second because the velocity is positive for $0 < t < \frac{1}{2}$. When the velocity is 0, the diver has reached the maximum height of the dive.

Figure 2.21

TEC Visual 3.1 uses the slope-a-scope to illustrate this formula.

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

Thus the exponential function $f(x) = e^x$ has the property that it is its own derivative. The geometrical significance of this fact is that the slope of a tangent line to the curve $y = e^x$ is equal to the y-coordinate of the point (see Figure 7).

EXAMPLE 8 If $f(x) = e^x - x$, find f' and f'' . Compare the graphs of f and f' .

SOLUTION Using the Difference Rule, we have

$$f'(x) = \frac{d}{dx}(e^x - x) = \frac{d}{dx}(e^x) - \frac{d}{dx}(x) = e^x - 1$$

In Section 2.8 we defined the second derivative as the derivative of f' , so

$$f''(x) = \frac{d}{dx}(e^x - 1) = \frac{d}{dx}(e^x) - \frac{d}{dx}(1) = e^x$$

The function f and its derivative f' are graphed in Figure 8. Notice that f has a horizontal tangent when $x = 0$; this corresponds to the fact that $f'(0) = 0$. Notice also that, for $x > 0$, $f'(x)$ is positive and f is increasing. When $x < 0$, $f'(x)$ is negative and f is decreasing.

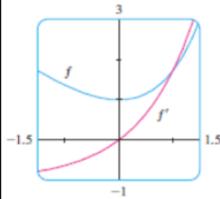


FIGURE 8

EXAMPLE 9 At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

SOLUTION Since $y = e^x$, we have $y' = e^x$. Let the x -coordinate of the point in question be a . Then the slope of the tangent line at that point is e^a . This tangent line will be parallel to the line $y = 2x$ if it has the same slope, that is, 2. Equating slopes, we get

$$e^a = 2 \quad a = \ln 2$$

Therefore the required point is $(a, e^a) = (\ln 2, 2)$. (See Figure 9.)

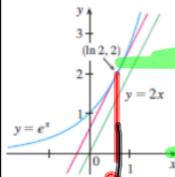


FIGURE 9

Handwritten notes: $\ln 2$ and $\ln 2$ with arrows pointing to the x and y coordinates of the intersection point in Figure 9.

Handwritten notes: "Let this point be a " and $y = e^x \Rightarrow y' = e^x \Rightarrow y'(a) = e^a$

Handwritten equation: $\ln e^a = \ln 2$

Handwritten equation: $a = \ln e^a = \ln 2$

Handwritten boxed equation: $a = \ln 2$

Handwritten circled text: "Ex."

Handwritten text: "1. Find the equation of the tangent line"

Handwritten text: "2. Sketch (on line)"

Handwritten text: $e^x, 2x, \text{ tangent } \ln 2$

Handwritten point-slope form equation: $y - y_0 = m(x - \frac{1}{a})$ with $m = e^{\ln 2}$

THEOREM 2.7 The Product Rule

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

Ex.

$$h(x) = (3x - 2x^2)(5 + 4x).$$

$$h(x) = 15x + 12x^2 - 10x^2 - 8x^3$$

$$h(x) = 15x + 2x^2 - 8x^3$$

$$h'(x) = 15 + 4x - 24x^2$$

$$h(x) = \underbrace{(3x - 2x^2)}_f \cdot \underbrace{(5 + 4x)}_g$$

$$h' = f'g + fg'$$

$$= (3 - 4x) \cdot (5 + 4x) + (3x - 2x^2) \cdot 4$$

$$= 15 + 12x - 20x - 16x^2 + 12x - 8x^2$$

$$= -24x^2 + 4x + 15$$

ex. $y = \frac{1}{x}$

Power Rule
 $y = \frac{1}{x} = x^{-1}$
 $y' = -1 \cdot x^{-2} = -\frac{1}{x^2}$

$y = \frac{1}{x}$
 $y' = \frac{0 \cdot x - 1 \cdot 1}{x^2} = -\frac{1}{x^2}$

ex. $y = \frac{5x - 2}{x^2 + 1} = \frac{f}{g}$

~~WRWB~~
 ~~$-10x^2 + 1$~~
 ~~$-2x \cdot 5x - 1$~~

$y' = \frac{f'g - fg'}{g^2} = \frac{5(x^2 + 1) - 2x(5x - 1)}{(x^2 + 1)^2}$

$= \frac{\cancel{5x^2} + 5 - \cancel{10x^2} + 2x}{(x^2 + 1)^2} = \frac{-5x^2 + 5 + 2x}{(x^2 + 1)^2}$

$$\frac{d}{dx}(\sin x) = \cos x.$$

$$\frac{d}{dx}(\cos x) = -\sin x.$$

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cdot \cos x - (-\sin x) \cdot \sin x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

$$\sec^2 x = \frac{1}{\cos^2 x}$$

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x.$$

$$\operatorname{cosec}^2 x = \frac{1}{\sin^2 x}$$

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$= \frac{0 \cdot \cos x - (-\sin x) \cdot 1}{\cos^2 x} = \frac{\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \cdot \sec x$$

$$(\sec x)' = \frac{\sin x}{\cos^2 x} = \sec x \cdot \tan x$$

$$y = x \cdot \sec x$$

$$y' = 1 \cdot \sec x + x \cdot \tan x \cdot \sec x$$

$$= \sec x (1 + x \cdot \tan x)$$

Chain Rule

$$y = x^2 + 2x \quad | \quad y = (x^2 + 2x)^{1/2}$$
$$y = \sin x \quad | \quad y = \sin^2 x, \quad y = \sin(x^2)$$

$$y = \sin^2 x$$
$$y = \sin x \cdot \sin x$$

$$y = \sin(x^2)$$
$$y = \sin(x \cdot x)$$

$$y = \cos x$$

$$y = \cos(5x)$$

$$y = e^x$$

$$y = e^{x^2}, \quad y = e^{2x}, \quad y = e^{\sin x}$$

$$y = \tan x$$

$$y = \tan(e^x)$$

$$y^{(u)} = F(u) = f(g(x))$$

↖ outer ↗ inner

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

= (the derivative of the outer) · (the derivative of the inner)

$$y = u^5, \quad u = x^2 + 1 \Rightarrow y = (x^2 + 1)^5$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2x \cdot 5 \cdot (x^2 + 1)^4$$

$5 \cdot u^4 = 5 \cdot (x^2 + 1)^4$

$$= 10x(x^2 + 1)^4$$

$$y = (x^3 - x)^{1/2} \Rightarrow y' = \frac{1}{2} (3x^2 - 1)(3x^2 - 1)^{-1/2}$$

$$y = \sin(3x) \Rightarrow y' = 3 \cdot \cos(3x)$$

$$y = e^{\sin x} = \exp(\sin x)$$

$$y' = \cos x \cdot e^{\sin x}$$

The simple power rule:

$$(x^n)' = n x^{n-1}$$

The general power rule:

$$[U(x)]^n = n(U(x))^{n-1} \cdot U'$$

ex $y(x) = \sqrt{x^2 + 1}$
 $= (x^2 + 1)^{1/2}$

$$y' = \frac{1}{2} (x^2 + 1)^{-1/2} \cdot 2x$$

Outer derivative inner derivative

$$y' = \frac{x}{(x^2 + 1)^{1/2}} = \frac{x}{\sqrt{x^2 + 1}}$$

ex

$$y = (x^2 - 3x + 2)^{3/2}$$

$$y' = \frac{3}{2} (x^2 - 3x + 2)^{1/2} \cdot (2x - 3)$$

$$= \frac{6x - 9}{2} \sqrt{x^2 - 3x + 2}$$

Example 7 – Simplifying by Factoring Out the Least Powers

Find the derivative of $f(x) = x^2\sqrt{1-x^2}$.

Solution:

$$f(x) = x^2\sqrt{1-x^2}$$

Write original function.

$$= x^2(1-x^2)^{1/2}$$

Rewrite.

$$f'(x) = x^2 \frac{d}{dx} [(1-x^2)^{1/2}] + (1-x^2)^{1/2} \frac{d}{dx} [x^2]$$

Product Rule

$$= x^2 \left[\frac{1}{2} (1-x^2)^{-1/2} (-2x) \right] + (1-x^2)^{1/2} (2x)$$

General Power Rule

$$= -x^3(1-x^2)^{-1/2} + 2x(1-x^2)^{1/2}$$

Simplify.

$$= x(1-x^2)^{-1/2} [-x^2(1) + 2(1-x^2)]$$

Factor.

$$= \frac{x(2-3x^2)}{\sqrt{1-x^2}}$$

Simplify.

Handwritten notes in red ink:

- Under the first term of the second line: $x(1-x^2)^{-1/2}$
- Under the second term of the second line: $2x(1-x^2)^{1/2}$
- Under the third line: $-x^2 - 2x^2 + 2$
- Under the fourth line: $x^2 - 2x^2 + 2$
- Under the fifth line: $2x(1-x^2)$
- Under the sixth line: $x(1-x^2)^{-1/2}$
- On the right side: $\frac{1}{2}(-2x) = -x$

$$(\sin x)' = \cos x$$

$$(\sin u)' = \cos u \cdot u'$$

$$(\cos x)' = -\sin x$$

$$(\cos u)' = -\sin u \cdot u'$$

$$(\tan x)' = \sec^2 x$$

$$(\tan u)' = \sec^2 u \cdot u'$$

$$(\cot x)' = -\csc^2 x$$

$$(\cot u)' = -\csc^2 u \cdot u'$$

$$(\sin 2x)' = \cos 2x \cdot 2$$

$$\begin{aligned} (\sin e^x)' &= (\cos e^x) \cdot e^x \\ &= \cos(x \cdot e^x) \end{aligned}$$

$$(\cos(x^2 + 5))' = -\sin(x^2 + 5) \cdot 2x$$

$$(\tan(x^2))' = \sec^2(x^2) \cdot 2x$$

$$\begin{aligned} (\cos(\sin x))' &= \\ &= -\sin(\sin x) \cdot \cos x \end{aligned}$$

ex

$$xy = 1 \Rightarrow y = \frac{1}{x}$$

$$\Rightarrow y' = -\frac{1}{x^2}$$

ex

$$xy^3 - \sin xy^2 + \sqrt{y} = 0$$

(implicit differential)

ex

$$y = x^2 \Rightarrow y' = 2x$$

$$y = 2^x \Rightarrow y' = 2^x \ln 2$$

$$y = x^x \Rightarrow y' = x^x \cdot x^{-1}$$

(logarithmic dif.f.)

Ex $\frac{d}{dx} (3x^3 - x) = 9x^2 - 1.$

degree

$$\frac{d}{dx} (y^2) = \frac{d}{dx} (y^2)$$

$$= 2y \cdot y'$$

$$\frac{d}{dx} (x + 3y^2) = 1 + 3 \cdot y'$$

→ power 2
1 · y⁰ · y'

$$\frac{d}{dx} (3y^2 + x^2 \cdot y^3)$$

(y²)
(3x²+2)²

$$= 6y \cdot y' + 2xy^3 + x^2 \cdot 3y^2 \cdot y'$$

≠ Sin(x²)

$$\frac{d}{dx} (\underbrace{\sin^2 x}_{(\sin x)^2} + \cos(y^2) + xy)$$

$$\Rightarrow \underbrace{2 \sin x \cdot \cos x}_{\text{outer}} + \underbrace{(-\sin y^2)}_{\text{inner}} \cdot 2y \cdot y' + 1 \cdot y + x \cdot y'$$

Find $\frac{dy}{dx}$ given that $y^3 + y^2 - 5y - x^2 = -4$.

Step 1: $3y^2 \cdot y' + 2y \cdot y' - 5y' - 2x = 0$

Step 2: $3y^2 y' + 2y y' - 5y' = 2x$

Step 3: $y' (3y^2 + 2y - 5) = 2x$

Step 4:
$$y' = \frac{2x}{3y^2 + 2y - 5}$$

Determine the slope of the graph of $3(x^2 + y^2)^2 = 100xy$ at the point $(3, 1)$.

Step 1: $6(x^2 + y^2) \cdot (2x + 2yy')$
 $= 100(1 \cdot y + xy')$

$(y^2)' = 2 \cdot y \cdot y' = y'$

Step 2: $(6x^2 + 6y^2)(2x + 2yy') = 100y + 100xy'$

$12x^3 + 12x^2 y y' + 12x y^2 + 12y^3 y' = 100y + 100xy'$
 $12x^2 y y' + 12y^3 y' - 100xy' = 100y - 12x^3 - 12x^2 y$

Step 3: $y' (12x^2 y + 12y^3 - 100x) = 100y - 12x^3 - 12x^2 y$

Step 4: $y' = \frac{100y - 12x^3 - 12x^2 y}{12x^2 y + 12y^3 - 100x}$
 $x=3, y=1$

$y' = \frac{100 - 96 - 108}{108 + 12 - 300} = \frac{-102}{-180}$
 = simplify

Derivatives of Logarithmic Functions

In this section we use implicit differentiation to find the derivatives of the logarithmic functions $y = \log_b x$ and, in particular, the natural logarithmic function $y = \ln x$. [It can be proved that logarithmic functions are differentiable; this is certainly plausible from their graphs (see Figure 1.5.12).]

1

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

$$\frac{d}{dx}(\log_b u) = \frac{1}{u \ln b} \cdot u'$$

If we put $b = e$ in Formula 1, then the factor $\ln b$ on the right side becomes $\ln e = 1$ and we get the formula for the derivative of the natural logarithmic function $\log_e x = \ln x$:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot u'$$

EXAMPLE 1 Differentiate $y = \ln(x^3 + 1)$.

SOLUTION To use the Chain Rule, we let $u = x^3 + 1$. Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{u} \frac{du}{dx}$$

$$= \frac{1}{x^3 + 1} (3x^2) = \frac{3x^2}{x^3 + 1}$$

$$3x^2 \cdot \frac{1}{x^3 + 1} = \frac{3x^2}{x^3 + 1}$$

$$\boxed{\frac{d}{dx} (\ln y) = \frac{y'}{y}}$$

EXAMPLE 2 Find $\frac{d}{dx} \ln(\sin x)$.

$$(\ln(u))' = \frac{u'}{u}, \quad u = \sin x$$

SOLUTION Using (3), we have

$$\frac{d}{dx} \ln(\sin x) = \frac{1}{\sin x} \frac{d}{dx} (\sin x) = \frac{1}{\sin x} \cos x = \cot x \quad \blacksquare$$

EXAMPLE 3 Differentiate $f(x) = \sqrt{\ln x}$.

SOLUTION This time the logarithm is the inner function, so the Chain Rule gives

$$f'(x) = \frac{1}{2}(\ln x)^{-1/2} \frac{d}{dx} (\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}} \quad \blacksquare$$

$$\frac{d}{dx} (\ln x)^{1/2} = \frac{1}{2} (\ln x)^{-1/2} \cdot \frac{1}{x}$$

EXAMPLE 4 Differentiate $f(x) = \log_{10}(2 + \sin x)$.

SOLUTION Using Formula 1 with $b = 10$, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx} \log_{10}(2 + \sin x) \\ &= \frac{1}{(2 + \sin x) \ln 10} \frac{d}{dx} (2 + \sin x) \\ &= \frac{\cos x}{(2 + \sin x) \ln 10} \end{aligned}$$

EXAMPLE 5 Find $\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}}$.

SOLUTION 1

$$\begin{aligned} \frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}} &= \frac{1}{\frac{x+1}{\sqrt{x-2}}} \frac{d}{dx} \frac{x+1}{\sqrt{x-2}} \quad \left\{ \begin{array}{l} \text{quotient} \\ \text{rule} \end{array} \right. \\ &= \frac{\sqrt{x-2}}{x+1} \frac{\sqrt{x-2} \cdot 1 - (x+1)(\frac{1}{2})(x-2)^{-1/2}}{x-2} \\ &= \frac{x-2 - \frac{1}{2}(x+1)}{(x+1)(x-2)} \\ &= \frac{x-5}{2(x+1)(x-2)} \end{aligned}$$

Handwritten notes: $u = \frac{x+1}{\sqrt{x-2}} \Rightarrow \frac{1}{u} \cdot u'$

Handwritten notes: $\frac{d}{dx} (x-2)^{-1/2} = -\frac{1}{2} \cdot (x-2)^{-3/2}$

Handwritten note: Find answer

example Find $\frac{dy}{dx}$:

$$e^{xy} + x^2 + y^2 + x^3 y^3 = 0$$

$$S1) e^{xy} \cdot (1 \cdot y + x \cdot y') + 2x + 2y \cdot y' + 3x^2 \cdot y^3 + x^3 \cdot 3y^2 \cdot y' = 0$$

$$S2) y' (x e^{xy} + 2y + 3x^2 y^2) = -y e^{xy} - 2x - 3x^2 y^3$$

$$S4) y' = \frac{-y e^{xy} - 2x - 3x^2 y^3}{x e^{xy} + 2y + 3x^2 y^2}$$

Common mistake:

$$(y^2)' = 2y \quad \times$$

$$(y^2)' = 2y \cdot y' \quad \checkmark$$

Recall : $\log_a(x \cdot y) = \log_a x + \log_a y$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a x^b = b \cdot \log_a x$$

$$\log_e x = \ln x.$$

Logarithmic differentiation

example : $y = x^x$

S1) $\ln y = \ln x^x$

S2) $\ln y = x \cdot \ln x$

S3) Differentiate

$$\frac{y'}{y} \stackrel{\text{Product Rule}}{=} 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$\frac{y'}{y} = (\ln x + 1)$$

$$y' = y (\ln x + 1)$$

$$y' = x^x (\ln x + 1)$$

example

$$y = x^{\sqrt{x}}$$

$$S1) \ln y = \ln x^{\sqrt{x}}$$

$$S2) \ln y = \frac{\sqrt{x} \cdot \ln x}{x^{1/2}}$$

$$S3) \frac{y'}{y} = \frac{1}{2} \left(\frac{-1/2}{x} \cdot \ln x + x^{-1/2} \cdot \frac{1}{x} \right)$$

$$\frac{y'}{y} = \left(\frac{1}{2\sqrt{x}} \cdot \ln x \right) + \frac{1}{\sqrt{x}}$$

$$y' = y \left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \right)$$

$$y' = x^{\sqrt{x}} \left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \right)$$

example $y = x^{\sin^2 x}$

S1) $\ln y = \ln x^{\sin^2 x}$

S2) $\ln y = \frac{\sin^2 x}{(\sin x)^2} \cdot \ln x$

S3) $\frac{y'}{y} = \underline{2 \sin x \cdot \cos x \cdot \ln x} + \sin^2 x \cdot \frac{1}{x}$

$$y' = y \left(2 \sin x \cos x \ln x + \frac{\sin^2 x}{x} \right)$$

$$y' = x^{\sin^2 x} \left(2 \sin x \cos x \ln x + \frac{\sin^2 x}{x} \right)$$

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called logarithmic differentiation.

EXAMPLE 7 Differentiate $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$. (ex: Use quotient rule)

SOLUTION We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \frac{3}{4} \ln x + \frac{1}{2} \ln(x^2 + 1) - 5 \ln(3x + 2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y} \frac{dy}{dx} = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x^2+1} - 5 \cdot \frac{3}{3x+2}$$

Solving for dy/dx , we get

$$\frac{dy}{dx} = y \left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2} \right)$$



$$y = \frac{x^{3/4} (x^2+1)^{1/2}}{(3x+2)^5}$$

S1) $\ln y = \ln \left(\frac{x^{3/4} (x^2+1)^{1/2}}{(3x+2)^5} \right)$

S2) $\ln y = \ln(x^{3/4} \cdot (x^2+1)^{1/2}) - \ln(3x+2)^5$

$\ln y = \ln(x^{3/4}) + \ln(x^2+1)^{1/2} - \ln(3x+2)^5$
 (only preCalculus till here)

S3) $\ln y = \frac{3}{4} \ln x + \frac{1}{2} \ln(x^2+1) - 5 \ln(3x+2)$

$$\frac{y'}{y} = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x^2+1} - 5 \cdot \frac{3}{3x+2}$$

$$y' = y \left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2} \right)$$

$$y' = \frac{x^{3/4} (x^2+1)^{1/2}}{(3x+2)^5} \left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2} \right)$$

Recall: $(\ln y)' = \frac{y'}{y}$ $\ln(3x^2-1) = \frac{6x}{3x^2-1}$

Logarithmic differentiation

Recall : $\log_b(x \cdot y) = \log_b x + \log_b y$.

$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

$\log_b x^r = r \cdot \log_b x$

$\log_e x := \ln x$

$y = x^2$, $y = 2^x$, $y = x^x$

Variable (under x in x^2), *Variable* (under x in 2^x), $y = x^x$ (circled)

$y' = 2x$, $y' = 2^x \ln 2$?

example : $y = x^x$

Recall: $(\ln y)' = \frac{y'}{y} \left\{ \ln(x^3+1) = \frac{3x^2}{x^3+1} \right\}$

ex. $y = x^x$.

S1) $\ln y = \ln x^x$

S2) $\ln y = x \cdot \ln x$

S3) Differentiate

$\frac{y'}{y} = \ln x + x \cdot \frac{1}{x}$

$\frac{y'}{y} = \ln x + 1$

$y' = y(\ln x + 1)$

$y' = x^x \cdot (\ln x + 1)$

ex. $y = x^{\sqrt{x}}$

S1) $\ln y = \ln x$

S2) $\ln y = \sqrt{x} \cdot \ln x$
 $x^{1/2}$

S3) Differentiate

$$\frac{y'}{y} = \frac{1}{2} x^{-1/2} \ln x + x^{1/2} \cdot \frac{1}{x}$$
$$y' = y \left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \right)$$

$$y' = \sqrt{x} \left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \right)$$

ex: $y = (\ln x)^{\sin x}$

S1) $\ln y = \ln(\ln x)^{\sin x}$

S2) $\ln y = \sin x \cdot \ln(\ln x)$

S3) Differentiate

$$\frac{y'}{y} = \frac{\cos x \cdot \ln(\ln x) + \sin x \cdot \frac{1}{x}}{\ln x} \quad \left| \frac{\ln(y)}{\ln x} \right|$$

$$\frac{y'}{y} = \cos x \cdot \ln(\sin x) + \sin x \cdot \frac{1}{x} \cdot \ln x$$

$$y' = y \left(\cos x \cdot \ln(\sin x) + \sin x \cdot \frac{1}{x} \right)$$

$(\ln x)^{\sin x}$

Common mistake: $y = x^x$

- $y = x \cdot x^{x-1}$ ~~X~~
- $\ln(\ln x) = \ln^2 x$ ~~X~~
- $\ln(\ln x) \neq \ln x \cdot \ln x$

ex. $y = \frac{x^{3/4} \sqrt{x^2+1}}{(3x+2)^5}$

a) Differentiate ✓

b) Differentiate by logarithmic diff

S1) $\ln y = \ln \left[\frac{x^{3/4} (x^2+1)^{1/2}}{(3x+2)^5} \right]$

S2) $\ln y = \ln x^{3/4} + \ln (x^2+1)^{1/2} - \ln (3x+2)^5$

$\ln y = \frac{3}{4} \ln x + \frac{1}{2} \ln(x^2+1) - 5 \ln(3x+2)$

S3) Differentiate

$\frac{y'}{y} = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \frac{2x}{x^2+1} - 5 \frac{3}{3x+2}$

$y' = y \left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2} \right)$

$y' = \frac{x^{3/4} (x^2+1)^{1/2}}{(3x+2)^5} \left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2} \right)$

Finding Related Rates

you can differentiate implicitly with respect to t to obtain the **related-rate** equation

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{\pi}{3}r^2h\right] \quad r = r(t), \quad h = h(t)$$

$$\frac{dV}{dt} = \frac{\pi}{3} \left[r^2 \frac{dh}{dt} + h \left(2r \frac{dr}{dt} \right) \right]$$

Differentiate with respect to t .

$$= \frac{\pi}{3} \left(r^2 \frac{dh}{dt} + 2rh \frac{dr}{dt} \right).$$

From this equation, you can see that the rate of change of V is related to the rates of change of both h and r .

$$\frac{dV}{dt} = \frac{\pi}{3} \left(2r \frac{dr}{dt} h(t) + r^2 \frac{dh}{dt} \right)$$



Example 1 – Two Rates That Are Related

Suppose x and y are both differentiable functions of t and are related by the equation $y = x^2 + 3$.

Find dy/dt when $x = 1$, given that $dx/dt = 2$ when $x = 1$.

Solution:

Using the Chain Rule, you can differentiate both sides of the equation *with respect to t* .

$$y = x^2 + 3$$

Write original equation.

$$\frac{d}{dt}[y] = \frac{d}{dt}[x^2 + 3]$$

Differentiate with respect to t .

$$\frac{dy}{dt} = 2x \frac{dx}{dt}$$

Chain Rule

When $x = 1$ and $dx/dt = 2$, you have $\frac{dy}{dt} = 2(1)(2) = 4$.

Given: $y(t) = x^2(t) + 3$
When $x=1$, $\frac{dx}{dt} = 2$

Ask for: $\frac{dy}{dt}$ when $x=1$

$$y = x^2 + 3$$

$$\frac{dy}{dt} = 2x(t) \cdot x'(t)$$

$$t=1 \Rightarrow \frac{dy}{dt} = 2 \cdot 1 \cdot 2 = 4$$

Example 2 – Ripples in a Pond

A pebble is dropped in a calm pond, causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

$$r = 4$$
$$\frac{dA}{dt} = ?$$

when

$$r = 4$$



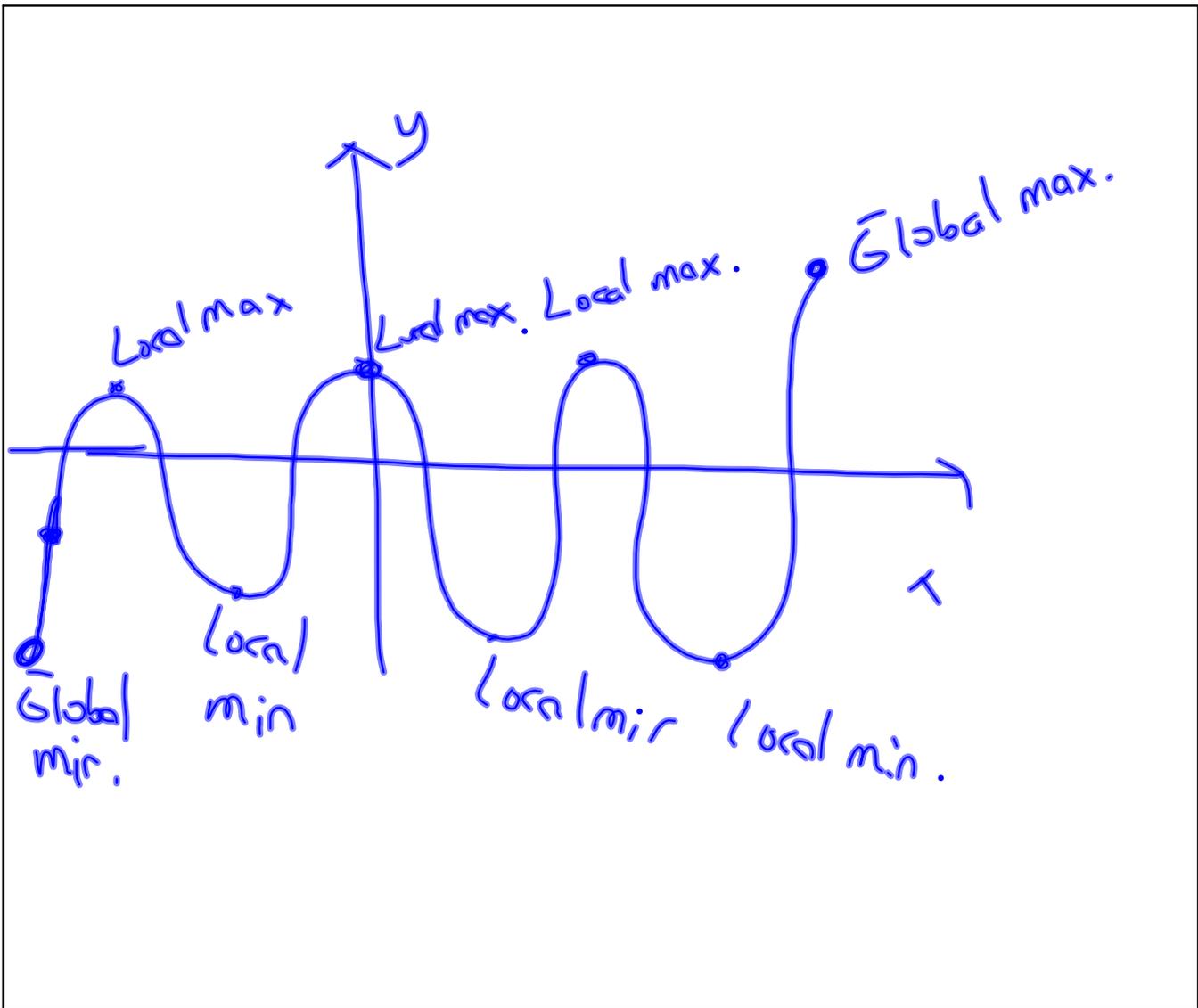
$$\frac{dr}{dt} = 1$$

$$A = \pi \cdot r^2$$

$$\frac{dA}{dt} = \pi \cdot \underbrace{2r}_{\text{outer}} \cdot \underbrace{r'}_{\text{inner}}$$

$$= \pi \cdot 2 \cdot 4 \cdot 1$$

$$= 8\pi$$



ex. Find the critical points of f

$$f(x) = x^{3/5} (4-x)$$

$$f'(x) = \frac{3}{5} x^{-2/5} (4-x) - 1 \cdot x^{3/5}$$

$$= \frac{3}{5 x^{2/5}} (4-x) - \frac{x^{3/5}}{1}$$

$$= \frac{3(4-x) - 5 \cdot x}{5 x^{2/5}} = \frac{12-8x}{5 x^{2/5}} = 0$$

at $x=0$, f' does not exist.

$$12-8x=0 \Rightarrow 12=8x \Rightarrow x = \frac{12}{8} = \frac{3}{2}$$

EXAMPLE 8 Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \quad \left(-\frac{1}{2} \leq x \leq 4\right) \quad -\frac{1}{2} \leq x \leq 4$$

$$f(x) = x^3 - 3x^2 + 1$$

$$1) f' = 3x^2 - 6x$$

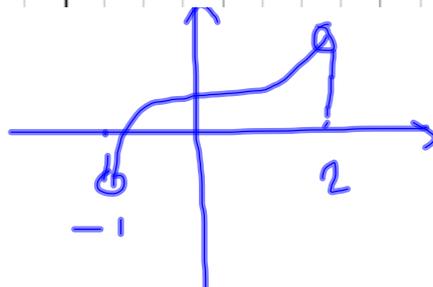
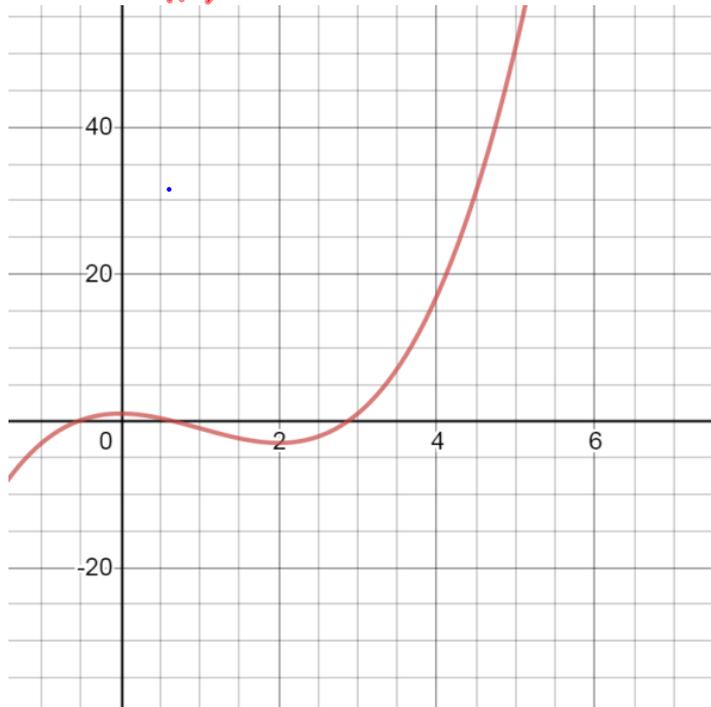
$$2) f' = 0 \Rightarrow 3x(x-2) = 0$$

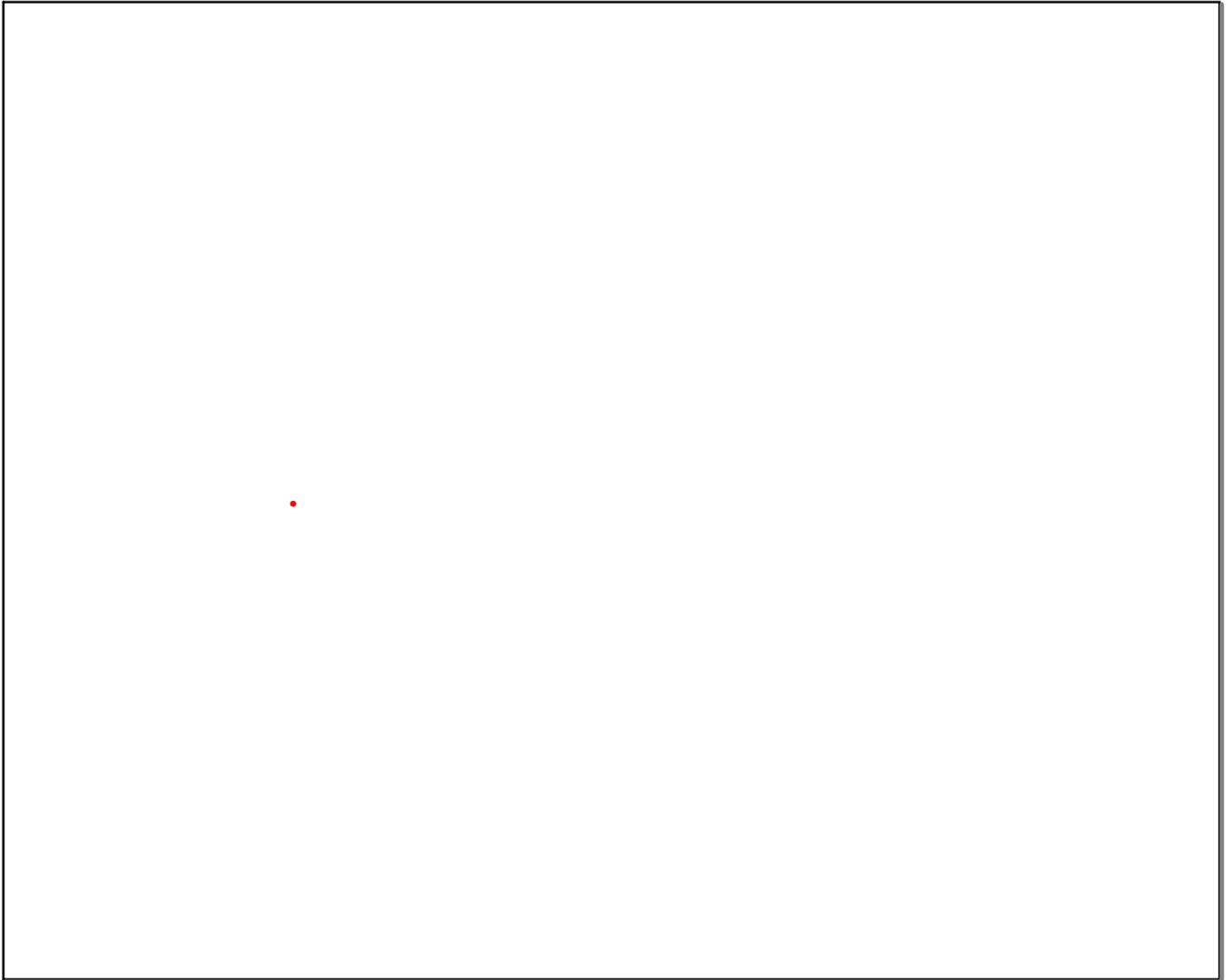
$$\begin{array}{c} \text{---} \\ | \\ -1/2 \end{array} \quad \begin{array}{c} \text{---} \\ | \\ 4 \end{array}$$

$$x=0 \quad x=2$$

x	0	2	-1/2	4
f(x)	4	-3	1/8	17

min. max.





Example 1 – Intervals on Which f Is Increasing or Decreasing

Find the open intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing or decreasing.

Solution:

Note that f is differentiable on the entire real number line and the derivative of f is

$$f(x) = x^3 - \frac{3}{2}x^2 \quad \text{Write original function.}$$

$$f'(x) = 3x^2 - 3x. \quad \text{Differentiate.}$$

$$f(x) = x^3 - \frac{3}{2}x^2$$

$$f'(x) = 3x^2 - \frac{3}{2} \cdot 2x^{2-1} = 3x^2 - 3x$$

$$3x^2 - 3x = 0 \Rightarrow 3x(x-1) = 0$$



$$f'(-1) = 6$$

$$f'(1/2) = \frac{3}{4} - \frac{3}{2} = -\frac{3}{4}$$

$$f'(2) = 12 - 6 = 6$$

$(-\infty, 0)$ and $(1, \infty)$ $f \nearrow$

$(0, 1)$ $f \searrow$

Example 2 – Applying the First Derivative Test

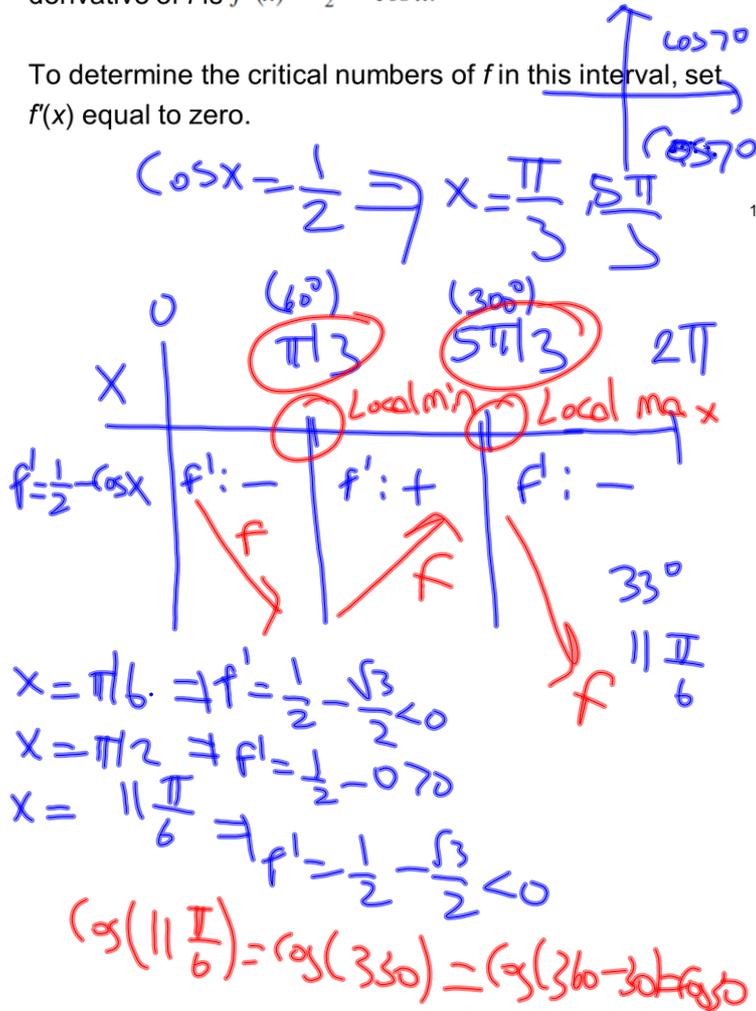
Find the relative extrema of the function $f(x) = \frac{1}{2}x - \sin x$ in the interval $(0, 2\pi)$.

Solution:

$$f' = \frac{1}{2} - \cos x = 0$$

Note that f is continuous on the interval $(0, 2\pi)$. The derivative of f is $f'(x) = \frac{1}{2} - \cos x$.

To determine the critical numbers of f in this interval, set $f'(x)$ equal to zero.



$x = \frac{\pi}{3} \Rightarrow \text{local min.}$
 $x = \frac{5\pi}{3} \Rightarrow \text{local max.}$

THEOREM 3.7 Test for Concavity

Let f be a function whose second derivative exists on an open interval I .

1. If $f''(x) > 0$ for all x in I , then the graph of f is concave upward on I .
2. If $f''(x) < 0$ for all x in I , then the graph of f is concave downward on I .

To apply Theorem 3.7, locate the x -values at which $f''(x) = 0$ or f'' does not exist. Use these x -values to determine test intervals. Finally, test the sign of $f''(x)$ in each of the test intervals.

Example 1 – Determining Concavity

Determine the open intervals on which the graph of

$$f(x) = \frac{6}{x^2 + 3}$$

is concave upward or downward.

Solution:

Begin by observing that f is continuous on the entire real line.

Next, find the second derivative of f .

$$f(x) = 6(x^2 + 3)^{-1} \quad \text{Rewrite original function.}$$

$$f' = -6(x^2 + 3)^{-2} \cdot 2x = -12x(x^2 + 3)^{-2}$$

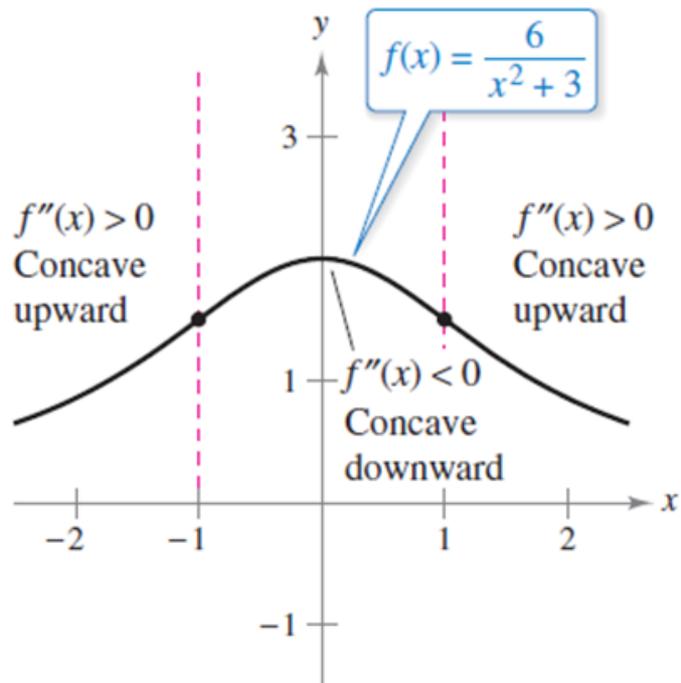
$$f'' = -12 \cdot (x^2 + 3)^{-2} - 12x \cdot -2 \cdot (x^2 + 3)^{-3} \cdot 2x$$
$$= -12(x^2 + 3)^{-2} + 48x^2(x^2 + 3)^{-3}$$

$$= -12(x^2 + 3)^{-3} (x^2 + 3 + 4x^2)$$

$$\stackrel{\text{Simplification}}{=} \frac{-12(3 - 3x^2)}{(x^2 + 3)^3}$$

$$f'' = 0 \quad f'' \text{ is undefined}$$
$$x^2 = 1 \Rightarrow x = \pm 1 \quad \text{No such values}$$

x	$-\infty$	-1	1	∞
Test Points	$x = -2$		$x = 0$	$x = 2$
f''	$f'' > 0$		$f'' < 0$	$f'' > 0$
Graph	U		n	U
Concavity	UP		down	UP



From the sign of f'' , you can determine the concavity of the graph of f .

Determine the points of inflection and discuss the concavity of the graph of $y = x^4 - 4x^3$

Solution: $f(2) = 16 - 32 = -16$

Differentiating twice produces the following.

$$y' = 4x^3 - 12x^2$$

$$y'' = 12x^2 - 24x$$

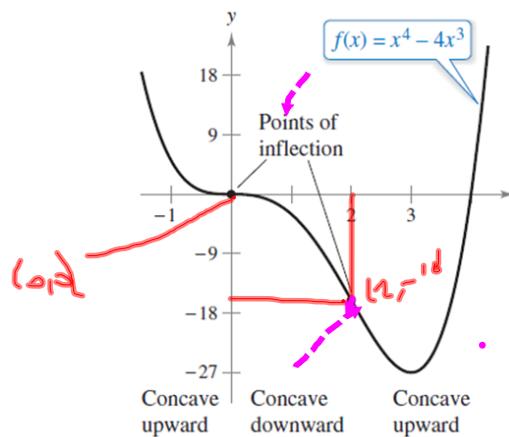
$$y'' = 12x(x - 2) = 0$$

$$\underline{x = 0} \quad \underline{x = 2}$$

x	$-\infty$	$x = 0$	$x = 2$	∞
test p:	$x = -1$	$x = 1$	$x = 3$	
f''	> 0	< 0	> 0	
f	U	n	U	

Concave up Concave down Concave up

$f(0) = 0$ $(0, 0)$ inflection point
 $f(2) = (2, -16)$ " point



Points of inflection can occur where $f''(x) = 0$ or f'' does not exist.

Example 4 – Using the Second Derivative Test

Find the extrema of $f(x) = -3x^5 + 5x^3$

Solution:

Begin by finding the critical numbers of f .

$$f'(x) = -15x^4 + 15x^2 = 15x^2(1 - x^2)$$

From this derivative, you can see that $x = -1, 0,$ and 1 are the only critical numbers of f .

By finding the second derivative

$$f''(x) = -60x^3 + 30x = 30x(1 - 2x^2)$$

you can apply the Second Derivative Test.

Example 4 – Solution

cont'd

Point	$(-1, -2)$	$(0, 0)$	$(1, 2)$
Sign of $f''(x)$	$f''(-1) > 0$	$f''(0) = 0$	$f''(1) < 0$
Conclusion	Relative minimum	Test fails	Relative maximum

Because the Second Derivative Test fails at $(0, 0)$, you can use the First Derivative Test and observe that f increases on the left and right of $x = 0$.

Apply first derivative test

EXAMPLE 5 Sketch a possible graph of a function f that satisfies the following conditions:

- (i) $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$
- (ii) $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2)$
- (iii) $\lim_{x \rightarrow -\infty} f(x) = -2$, $\lim_{x \rightarrow \infty} f(x) = 0$

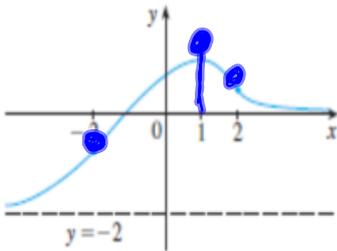


FIGURE 9

SOLUTION Condition (i) tells us that f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. Condition (ii) says that f is concave upward on $(-\infty, -2)$ and $(2, \infty)$, and concave downward on $(-2, 2)$. From condition (iii) we know that the graph of f has two horizontal asymptotes: $y = -2$ (to the left) and $y = 0$ (to the right).

We first draw the horizontal asymptote $y = -2$ as a dashed line (see Figure 9). We then draw the graph of f approaching this asymptote at the far left, increasing to its maximum point at $x = 1$, and decreasing toward the x -axis as at the far right. We also make sure that the graph has inflection points when $x = -2$ and 2 . Notice that we made the curve bend upward for $x < -2$ and $x > 2$, and bend downward when x is between -2 and 2 .

Local max. ■

x	$-\infty$	-2	1	2	∞
		inflection		inflection point	
f'	> 0 f: →	> 0 f: ↗	< 0 f: ↘	< 0 f: ↘	
f''	> 0 f: ↗ up	< 0 f: ↘ down	< 0 f: ↘ down	> 0 f: ↗ up	

$y = -2$, $y = 0$ are horizontal asymptotes. (x-axis)

Ex. $f(x) = x^3 + 6x^2 + 9x$.

d) Determine the domain?

$$D = (-\infty, \infty),$$

$$D = \mathbb{R}$$

b) Find asymptotes, if any.

Since f is a polynomial, no vertical and horizontal asymptotes.

c) Determine f is odd or even?

Is f symmetric.

$$f(x) = x^3 + 6x^2 + 9x$$

Recall: $f(-x) = f(x) \Rightarrow$ even \Rightarrow y-axis

$f(-x) = -f(x) \Rightarrow$ odd \Rightarrow origin symmetry

$$f(-x) = -x^3 + 6x^2 - 9x$$

$$\neq f(x)$$

$$\neq -f(x) = -x^3 - 6x^2 - 9x$$

f is neither odd nor even

d) Find x-intercept, y-intercept

y-intercept: Set $x = 0$

$$y = x^3 + 6x^2 + 9x \Rightarrow y = 0$$

~~$(0, 0)$ - y intercept.~~

x-intercept: Set $y = 0$

$$x^3 + 6x^2 + 9x = 0 \Rightarrow x(x^2 + 6x + 9) = 0$$

$$\Rightarrow x(x+3)^2 = 0 \Rightarrow x = 0, x = -3$$

~~$(0, 0), (-3, 0)$ - x intercepts.~~

e) Determine the interval of increase and decrease? Determine relative extrema, if any?

$$f = x^3 + 6x^2 + 9x$$

$$f' = 3x^2 + 12x + 9 = 3(x^2 + 4x + 3) = 0$$

$$= 3(x+3)(x+1) = 0$$

Critical points: $x = -3, x = -1$

x	$-\infty$ x=-4	-3	-1	∞ x=4
f'	>0	<0	>0	
f				

$x = -3$ relative max. $f(-3) = -27 + 54 - 27 = 0$
 $(-3, 0)$

$x = -1$ relative min. $f(-1) = -1 + 6 - 9 = -4$
 $(-1, -4)$

f) Determine the intervals of concave up and down? Determine inflection points if any?

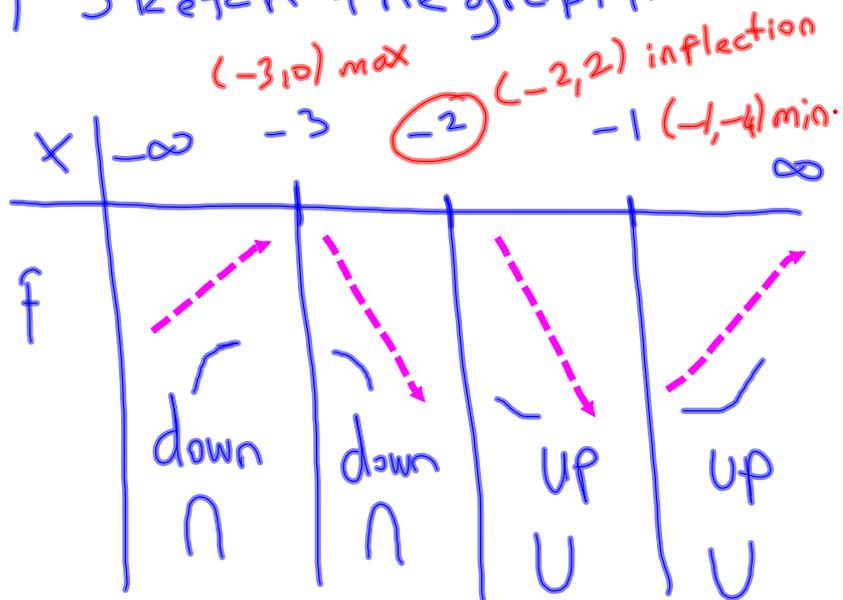
$$f'' = 6x + 12 = 0 \Rightarrow x = -2$$

x	(-5)	(-2) inflection point	(5)
f''	<0		>0
f	∩ down		∪ up

$$f(-2) = -8 + 24 - 18 = -2$$

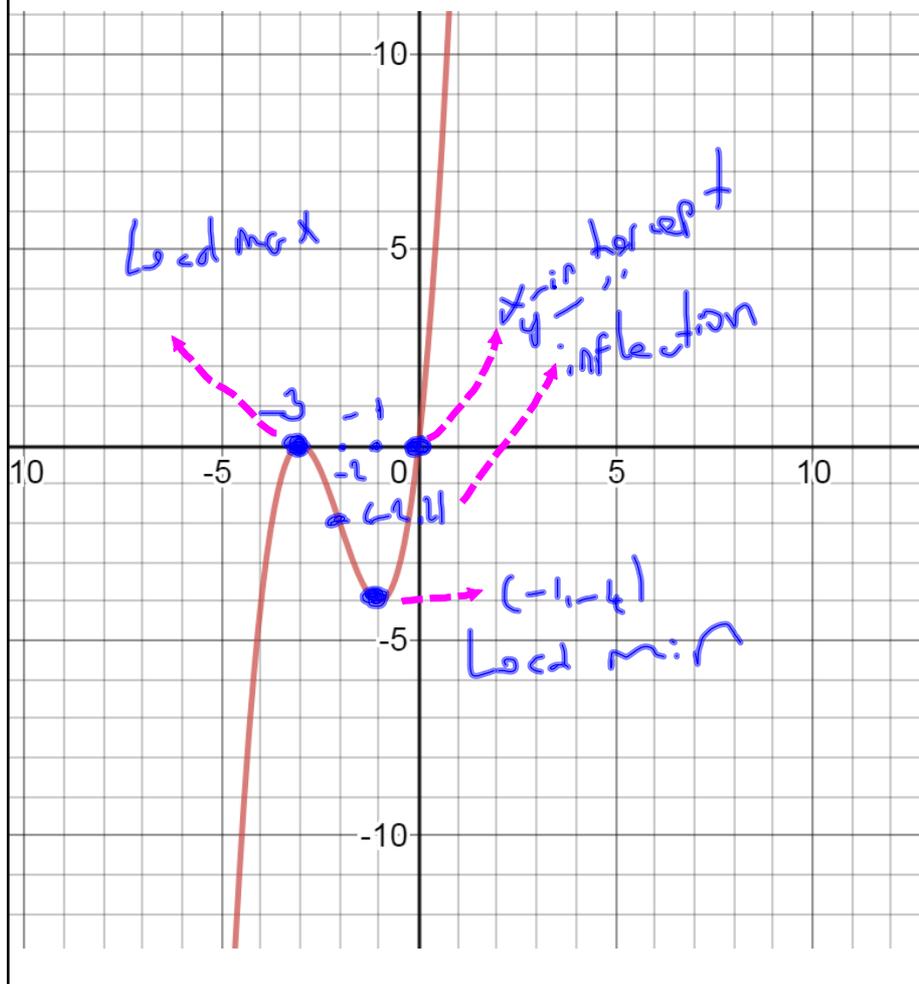
$(-2, -2)$ is inflection point

g) Sketch the graph.



$(0, 0)$, y -intercept

$(0, 0)$, $(-3, 0)$ x -intercept





[Example. Let $f(x) = \frac{x^2 - 9}{x^2 - 4}$.

a) Find the domain.

$$\mathbb{R} \setminus \{-2, 2\}$$

$$(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$$

b) Determine whether $f(x)$ is even or odd? Is there any symmetry

$$f(x) = \frac{x^2 - 9}{x^2 - 4}$$

$$f(-x) = \frac{(-x)^2 - 9}{(-x)^2 - 4} = \frac{x^2 - 9}{x^2 - 4} = f(x)$$

Since $f(-x) = f(x)$, $f(x)$ is even.

Its graph is symmetric wr.t y-axis

c) Find x-intercept and y-intercept, if any

y-intercept: Set $x=0$ $(0, 2.25)$

$$f(x) = \frac{x^2 - 9}{x^2 - 4} \Rightarrow f(0) = \frac{9}{4} = 2.25$$

x-intercept: Set $y=0$.

$$\frac{x^2 - 9}{x^2 - 4} = 0 \Rightarrow x = -3, 3$$

d) Find horizontal and vertical asymptotes: $f(x) = \frac{1 \cdot x^2 - 9}{1 \cdot x^2 - 4}$

$$\lim_{x \rightarrow \pm\infty} f(x) = 1 \quad y=1 \text{ is horizontal asymptote}$$

$$\lim_{x \rightarrow 2^+} f(x) = -\infty \quad \lim_{x \rightarrow 2^-} f(x) = +\infty$$

$$\lim_{x \rightarrow -2^+} f(x) = \infty \quad \lim_{x \rightarrow -2^-} f(x) = -\infty$$

$x=2, x=-2$ are vertical asymptotes

e) Find interval of decrease and increase and extremum points, if any.

$$f(x) = \frac{x^2 - 9}{x^2 - 4} \Rightarrow f' = \frac{2x(x^2 - 4) - 2x(x^2 - 9)}{(x^2 - 4)^2}$$

$$f' = \frac{10x}{(x^2 - 4)^2}$$

critical points: $0, -2, 2$
 $f'=0$ f' undefined

x	$-\infty$	-2	0	2	∞
	$x=-3$	$x=-1$	$x=1$	$x=3$	
y'	<0	<0	>0	>0	
y					

$x=0 \Rightarrow$ local minimum

$$f(0) = \frac{0^2 - 9}{0^2 - 4} = \frac{9}{4} \quad (0, \frac{9}{4}) = (0, 2.25) \text{ is local min.}$$

f) Find intervals of concave up and down and inflection points if any.

$$f' = \frac{10x}{(x^2-4)^2} \Rightarrow f'' = \frac{10(x^2-4)^2 - 2(x^2-4) \cdot 2x}{(x^2-4)^4}$$

$$= \frac{10(x^2-4) \left((x^2-4) - 4x^2 \right)}{(x^2-4)^3}$$

$$= \frac{10(-4-3x^2)}{(x^2-4)^3} = \frac{-10(3x^2+4)}{(x^2-4)^3}$$

f'' is never zero

but undefined when $x = -2, x = 2$

x	$-\infty$ $x = -3$	-2	2	∞ $x = 3$
y''	< 0		> 0	< 0
y	\cap up		\cup down	\cap up

No inflection points, because f is undefined at $x = -2, x = 2$

Summary :

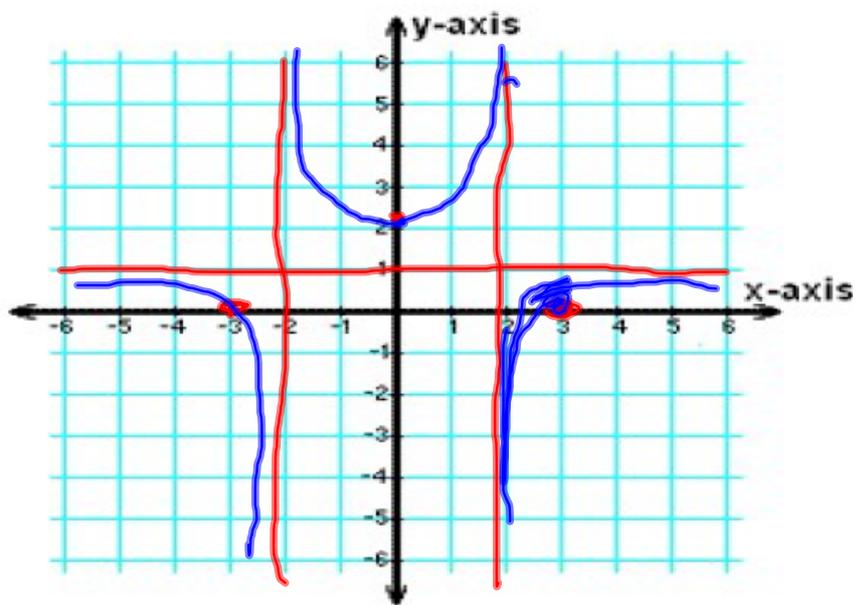
x	$-\infty$	-2	0	2	∞
y'	< 0	< 0	> 0	> 0	> 0
y''	< 0	> 0	> 0	< 0	< 0
y	 \cap	 \cup	 \cup	 \cap	

$x = -2, 2$ vertical asymptot

$y = 1$ horizontal asymptot

$$\lim_{x \rightarrow 2^+} f(x) = -2$$

$$\lim_{x \rightarrow 2^-} f(x) = +\infty$$



EXAMPLE 1 Find $\lim_{x \rightarrow 1} \frac{\ln x}{x-1}$. $= \frac{0}{0}$

SOLUTION Since

$$\lim_{x \rightarrow 1} \ln x = \ln 1 = 0 \quad \text{and} \quad \lim_{x \rightarrow 1} (x-1) = 0$$

the limit is an indeterminate form of type $\frac{0}{0}$, so we can apply l'Hospital's Rule:

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\ln x}{x-1} &= \lim_{x \rightarrow 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x \rightarrow 1} \frac{1/x}{1} = \frac{1/1}{1} = 1 \\ &= \lim_{x \rightarrow 1} \frac{1}{x} = 1 \end{aligned}$$

EXAMPLE 2 Calculate $\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$. $= \frac{\infty}{\infty}$

SOLUTION We have $\lim_{x \rightarrow \infty} e^x = \infty$ and $\lim_{x \rightarrow \infty} x^2 = \infty$, so the limit is an indeterminate form of type ∞/∞ , and l'Hospital's Rule gives

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^2)} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \frac{\infty}{\infty}$$

Since $e^x \rightarrow \infty$ and $2x \rightarrow \infty$ as $x \rightarrow \infty$, the limit on the right side is also indeterminate, but a second application of l'Hospital's Rule gives

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty$$

We applied l'Hospital's rule two times.

EXAMPLE 3 Calculate $\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}}$. $= \frac{\infty}{\infty}$

SOLUTION Since $\ln x \rightarrow \infty$ and $\sqrt{x} \rightarrow \infty$ as $x \rightarrow \infty$, l'Hospital's Rule applies:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x \rightarrow \infty} \frac{1/x}{1/(2\sqrt{x})} = \frac{\infty}{\infty}$$

Notice that the limit on the right side is now indeterminate of type $\frac{\infty}{\infty}$. But instead of applying l'Hospital's Rule a second time as we did in Example 2, we simplify the expression and see that a second application is unnecessary:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0$$

$$\frac{1}{\sqrt{x}} \cdot \frac{2\sqrt{x}}{1} = \frac{2}{\sqrt{x}}$$

EXAMPLE 4 Find $\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}$. (See Exercise 2.2.50.) $= \frac{0}{0} \checkmark$

SOLUTION Noting that both $\tan x - x \rightarrow 0$ and $x^3 \rightarrow 0$ as $x \rightarrow 0$, we use l'Hospital's Rule:

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} = \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2} = \frac{0}{0} \checkmark$$

Since the limit on the right side is still indeterminate of type $\frac{0}{0}$, we apply l'Hospital's Rule again:

$$\lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x}{6x} \quad \text{(l'Hospital's Rule)}$$

Because $\lim_{x \rightarrow 0} \sec^2 x = 1$, we simplify the calculation by writing

$$\lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x}{6x} = \frac{1}{3} \lim_{x \rightarrow 0} \sec^2 x \cdot \lim_{x \rightarrow 0} \frac{\tan x}{x} = \frac{1}{3} \lim_{x \rightarrow 0} \frac{\tan x}{x} \rightarrow \frac{0}{0}$$

We can evaluate this last limit either by using l'Hospital's Rule a third time or by writing $\tan x$ as $(\sin x)/(\cos x)$ and making use of our knowledge of trigonometric limits. Putting together all the steps, we get

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} &= \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x}{6x} \\ &= \frac{1}{3} \lim_{x \rightarrow 0} \frac{\tan x}{x} = \frac{1}{3} \lim_{x \rightarrow 0} \frac{\sec^2 x}{1} = \frac{1}{3} \end{aligned}$$

EXAMPLE 5 Find $\lim_{x \rightarrow \pi^-} \frac{\sin x}{1 - \cos x}$.

SOLUTION If we blindly attempted to use l'Hospital's Rule, we would get

$$\lim_{x \rightarrow \pi^-} \frac{\sin x}{1 - \cos x} = \lim_{x \rightarrow \pi^-} \frac{\cos x}{\sin x} = -\infty \quad \text{Wrong!}$$

This is **wrong!** Although the numerator $\sin x \rightarrow 0$ as $x \rightarrow \pi^-$, notice that the denominator $(1 - \cos x)$ does not approach 0, so l'Hospital's Rule can't be applied here.

The required limit is, in fact, easy to find because the function is continuous at π and the denominator is nonzero there:

$$\lim_{x \rightarrow \pi^-} \frac{\sin x}{1 - \cos x} = \frac{\sin \pi}{1 - \cos \pi} = \frac{0}{1 - (-1)} = 0$$

If $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = \infty$ (or $-\infty$), then it isn't clear what the value of $\lim_{x \rightarrow a} [f(x)g(x)]$, if any, will be. There is a struggle between f and g . If f wins, the answer will be 0; if g wins, the answer will be ∞ (or $-\infty$). Or there may be a compromise where the answer is a finite nonzero number. This kind of limit is called an **indeterminate form of type $0 \cdot \infty$** . We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{1/g} \quad \text{or} \quad fg = \frac{g}{1/f}$$

EXAMPLE 6 Evaluate $\lim_{x \rightarrow 0^+} x \ln x$. = $0 \cdot -\infty$

SOLUTION The given limit is indeterminate because, as $x \rightarrow 0^+$, the first factor (x) approaches 0 while the second factor ($\ln x$) approaches $-\infty$. Writing $x = 1/(1/x)$, we have $1/x \rightarrow \infty$ as $x \rightarrow 0^+$, so l'Hospital's Rule gives

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} (-x) = 0$$

NOTE In solving Example 6 another possible option would have been to write

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{x}{1/\ln x}$$

This gives an indeterminate form of the type $\frac{0}{0}$, but if we apply l'Hospital's Rule we get a more complicated expression than the one we started with. In general, when we rewrite an indeterminate product, we try to choose the option that leads to the simpler limit.

EXAMPLE 7 Compute $\lim_{x \rightarrow 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$. = $\infty - \infty$

SOLUTION First notice that $1/(\ln x) \rightarrow \infty$ and $1/(x-1) \rightarrow \infty$ as $x \rightarrow 1^+$, so the limit is indeterminate of type $\infty - \infty$. Here we can start with a common denominator:

$$\lim_{x \rightarrow 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) = \lim_{x \rightarrow 1^+} \frac{x-1 - \ln x}{(x-1) \ln x} = \frac{0}{0}$$

Both numerator and denominator have a limit of 0, so l'Hospital's Rule applies, giving

$$\lim_{x \rightarrow 1^+} \frac{x-1 - \ln x}{(x-1) \ln x} = \lim_{x \rightarrow 1^+} \frac{1 - \frac{1}{x}}{(x-1) \cdot \frac{1}{x} + \ln x} = \frac{0}{0}$$

Again we have an indeterminate limit of type $\frac{0}{0}$, so we apply l'Hospital's Rule a second time:

$$\begin{aligned} \lim_{x \rightarrow 1^+} \frac{x-1 - \ln x}{(x-1) \ln x} &= \lim_{x \rightarrow 1^+} \frac{1}{1 + x \cdot \frac{1}{x} + \ln x} \\ &= \lim_{x \rightarrow 1^+} \frac{1}{2 + \ln x} = \frac{1}{2} \end{aligned}$$

■ Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x \rightarrow a} [f(x)]^{g(x)}$$

1. $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$ type 0^0 ✓
2. $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = 0$ type ∞^0 ✓
3. $\lim_{x \rightarrow a} f(x) = 1$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$ type 1^\pm ✓

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{let } y = [f(x)]^{g(x)}, \text{ then } \ln y = g(x) \ln f(x)$$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$$

(Recall that both of these methods were used in differentiating such functions.) In either method we are led to the indeterminate product $g(x) \ln f(x)$, which is of type $0 \cdot \infty$.

EXAMPLE 9 Calculate $\lim_{x \rightarrow 0^+} (1 + \sin 4x)^{\cot x} = 1^{\infty}$

SOLUTION First notice that as $x \rightarrow 0^+$, we have $1 + \sin 4x \rightarrow 1$ and $\cot x \rightarrow \infty$, so the given limit is indeterminate (type 1^∞). Let

$$y = (1 + \sin 4x)^{\cot x}$$

$$\text{Then } \ln y = \ln[(1 + \sin 4x)^{\cot x}] = \cot x \cdot \ln(1 + \sin 4x) = \frac{\ln(1 + \sin 4x)}{\tan x} = \frac{0}{0}$$

so l'Hospital's Rule gives

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \frac{\ln(1 + \sin 4x)}{\tan x} = \lim_{x \rightarrow 0^+} \frac{\frac{4 \cos 4x}{1 + \sin 4x}}{\sec^2 x} = 4$$

So far we have computed the limit of $\ln y$, but what we want is the limit of y . To find this we use the fact that $y = e^{\ln y}$:

$$\lim_{x \rightarrow 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} e^{\ln y} = e^4$$

Handwritten notes for Example 9:

$$\lim_{x \rightarrow 0^+} \ln y = 4$$

$$\lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} e^{\ln y} = e^4$$

EXAMPLE 10 Find $\lim_{x \rightarrow 0^+} x^x$.

SOLUTION Notice that this limit is indeterminate since $0^x = 0$ for any $x > 0$ but $x^0 = 1$ for any $x \neq 0$. (Recall that 0^0 is undefined.) We could proceed as in Example 9 or by writing the function as an exponential:

$$\lim_{x \rightarrow 0^+} x^x = (e^{\ln x})^x = e^{x \ln x} = e^{\lim_{x \rightarrow 0^+} x \ln x}$$

In Example 6 we used l'Hospital's Rule to show that

$$\lim_{x \rightarrow 0^+} x \ln x = 0 = e^0 = 1$$

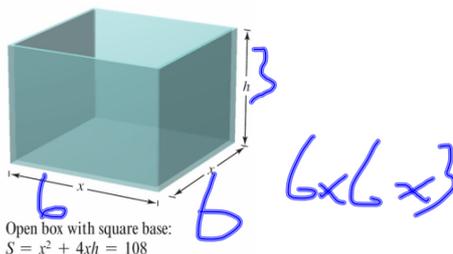
Therefore

$$\lim_{x \rightarrow 0^+} x^x = \lim_{x \rightarrow 0^+} e^{x \ln x} = e^0 = 1$$



Example 1 – Finding Maximum Volume

A manufacturer wants to design an open box having a square base and a surface area of 108 square inches, as shown in Figure 3.53. What dimensions will produce a box of maximum volume?



Example 1 – Solution

cont

Because V is to be maximized, you want to write V as a function of just one variable.

To do this, you can solve the equation $x^2 + 4xh = 108$ for h in terms of x to obtain $h = (108 - x^2)/(4x)$.

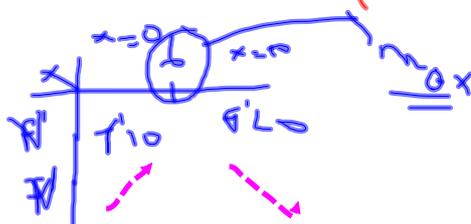
Substituting into the primary equation produces

$$\begin{aligned}
 V &= x^2h && \text{Function of two variables} \\
 &= x^2 \left(\frac{108 - x^2}{4x} \right) = 27x - \frac{x^3}{4} && \text{Substitute for } h. \\
 &= 27x - \frac{x^3}{4} = V(x) && \text{Function of one variable}
 \end{aligned}$$

$$V'(x) = 27 - \frac{3}{4}x^2 = 0$$

$$27 = \frac{3}{4}x^2$$

$$x^2 = 36 \Rightarrow x = 6$$



$$h = \frac{108 - x^2}{4x} = \frac{108 - 36}{24}$$

$$x = 6, h = 3 \quad V = \frac{72}{24} = 3$$

$$0 < x \leq \sqrt{108} \quad \sqrt{108} = 6 \quad \sqrt{108} = 6$$

2. Examples

- 1.) Find two numbers whose difference is 100 and whose product is a minimum.

What needs to be maximized or minimized. The *product* P needs to be minimized. If the two numbers are x and y , we can say that

$$P = xy$$

We need to eliminate a variable. To do that we use the fact that the difference between the two numbers has to be 100. This is the *constraint*:

$$x - y = 100$$

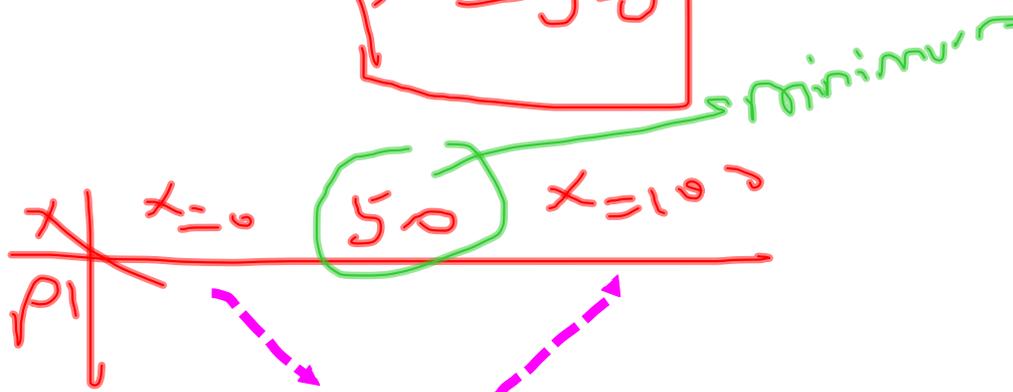
So we can say $y = x - 100$ and plug that into the equation for P ,

$$P = xy = x(x - 100) = x^2 - 100x$$

$$P = x(x - 100) = x^2 - 100x$$

$$P' = 2x - 100 = 0 \quad x \in \mathbb{R}$$

$$x = 50$$



$$x - 100 = y$$

$$50 - 100 = y$$

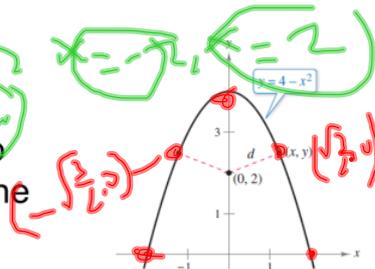
$$x = 50$$

$$y = -50$$

Which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$?

Solution:

Figure 3.55 shows that there are two points at a minimum distance from the point $(0, 2)$.



The quantity to be minimized is distance: $d = \sqrt{(x - 0)^2 + (y - 2)^2}$.

The distance between the point $(0, 2)$ and a point (x, y) on the graph of $y = 4 - x^2$ is given by

$$d = \sqrt{(x - 0)^2 + (y - 2)^2}$$

Primary equation

$$d = \sqrt{x^2 + (4 - x^2 - 2)^2}$$

$$= \sqrt{x^2 + x^4 - 4x^2 + 4}$$

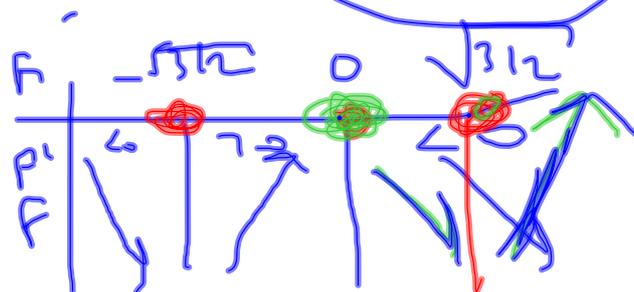
$$= \sqrt{x^4 - 3x^2 + 4}$$

Because d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of $f(x) = x^4 - 3x^2 + 4$.

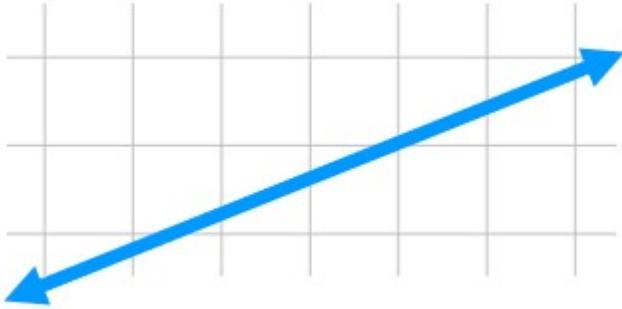
$$f' = 4x^3 - 6x = x(4x^2 - 6) = 0$$

$$(x=0) \quad 4x^2 - 6 = 0 \Rightarrow x^2 = \frac{6}{4}$$

$$x = \pm \sqrt{\frac{3}{2}}$$



Positive Slope

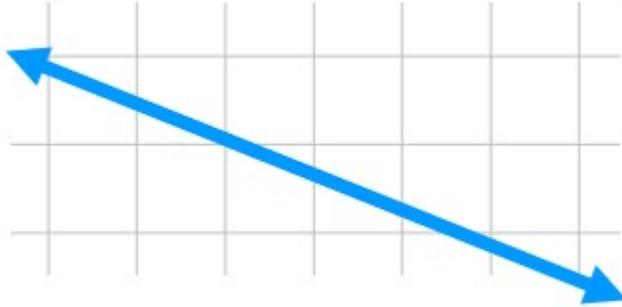


Line goes **UP**
from left to right.

y-values are **INCREASING** as
x-values are increasing

If the y-values are **increasing** as the x-values increase, the line has a positive slope. If you trace the line with your finger from left to right (the same order you read a book), the line will go **up** to the right.

Negative Slope



Line goes **DOWN**
from left to right.

y-values are **DECREASING**
as x-values are increasing

If the y-values are **decreasing**, the line has a **negative** slope. If you trace the line with your finger from left to right (the same direction that you read a book), the line will go down to the right.